Additional problems for *Optimization Models in Electricity Markets*

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Contents

Preface

This is a supporting manual for the textbook on *Optimization Models for Electricity Markets*. It includes problems that have been developed in the context of exams, projects, and course assignments for the following courses that I have offered at the Université catholique de Louvain in Belgium and the National Technical University of Athens in Greece in both Bachelors and Masters level:

- Quantitative Energy Economics (UCLouvain, masters)
- Operations Research (UCLouvain, masters)
- Power System Economics (NTUA, bachelors and masters)
- Power System Reliability (NTUA, bachelors)
- Mathematical Programming Models (NTUA, bachelors)
- Energy Economics (NTUA, bachelors and masters)

The material in this manual will be updated continuously. Solutions to the textbook exercises are available by the book publisher in a separate solutions manual.
There are three additional chapters in this supporting manual which treat the following topics:

- Dynamic programming (chapter 15)
- Integer programming (chapter 16)

Chapter 1

Introduction

Problem 1.1: Consider the system of paragraph 1.2 of the textbook, with the technology options that are indicated in table 2. The original load duration curve of the system is expressed as follows:

- Base load: 7086 MW, 8760 hours
- Medium load: 1918 MW, 7000 hours
- Peak load: 2165 MW, 1500 hours

Suppose that an ideal storage is introduced in the system, without losses and with unlimited technical capabilities (power rating and energy capacity), which results in a constant load throughout the year.

Question 1: Describe the new load duration curve.

Question 2: Describe the new optimal investment mix.

Question 3: Compute the new operating cost of the system and compare it to the long-run operating cost which occurs in the absence of storage. Is the new cost greater or lower, and why?

Question 4: Compute the long-run market equilibrium price for question 2.

Solution

Question 1: The new load duration curve is flat but has the same surface as the original one. Therefore its duration is 8760 hours and its height in MW is given by the solution of the following equation:

$$\Delta D = \frac{1500}{8760}2165 + \frac{7000}{8760}1918 + \frac{8760}{8760}7086 = 8989.4 \text{ MW}$$

Question 2: Because we have a single horizontal slice of duration 100%, we only choose nuclear for a capacity of 8989.4 MW.

Question 3: The new investment cost is computed as follows:

$$8989.4 \cdot 32 = 287660.8 \text{ \$/h}$$

The new operating cost is computed as

$$8989.4 \cdot 6.5 = 58431.4 \text{ \$/h}$$

The total operating cost is computed as the sum, 346091.9 \$/h.
The investment cost in the absence of storage is computed as

$$7086 \cdot 32 + 1918 \cdot 16 + 2165 \cdot 5 = 268265 \text{ \$/h}$$

The fuel cost in the absence of storage is computed as

$$7086 \cdot 6.5 \cdot 1 + 1918 \cdot 25 \cdot \frac{7000}{8760} + 2165 \cdot 80 \cdot \frac{1500}{8760} = 114032.7 \text{ \$/h}$$

The total cost is computed as the sum, 382297.7 \$/h. The cost is greater. Storage results in serving the same total amount of energy, but with lower-cost options, because it flattens the load profile.

Question 4: The profits from the energy market should exactly cover the investment cost of nuclear. Therefore

$$\mu_{nuc} = 32$$

And profitability is the difference between energy price and marginal cost, thus

$$\mu_{nuc} = \lambda - 6.5 \Rightarrow \lambda = 38.5 \text{ \$/MWh}$$

Chapter 2

Mathematical background

Problem 2.1 Implement the unit commitment model of example 2.3 in mathematical programming code, and confirm the primal optimal solution.

Solution: The code is available in the textbook website.

Problem 2.2: The following problem maximizes the social surplus from the trade of electric energy in an electricity auction:

$$\begin{aligned}
& \max_{p,d,u} 100d_A + 20d_B - 30p_C - 90p_D \\
& d_A \leq 90 \\
& d_B \leq 100 \\
& p_C = 100u_C \\
& p_D \leq 90 \\
(\lambda) : \quad & d_A + d_B - p_C - p_D = 0 \\
& d_A \geq 0, d_B \geq 0, p_D \geq 0, u_C \in \{0, 1\}
\end{aligned}$$

Question 1: What is the optimal solution of the model? What is the objective function value $p^\star$?

Question 2: The dual function from the relaxation of the market clearing constraint is expressed as follows:

$$\begin{aligned}
& g(\lambda) = \{\max_{p,u,d} 100d_A + 20d_B - 30p_C - 90p_D - \lambda(d_A + d_B - p_C - p_D), \\
& d_A \le 90, d_B \le 100, p_C = 100u_C, p_D \le 90 \\
& d_A \ge 0, d_B \ge 0, p_D \ge 0, u_C \in \{0, 1\}\} \\
& = g_A(\lambda) + g_B(\lambda) + g_C(\lambda) + g_D(\lambda)
\end{aligned}$$

where:

$$\begin{aligned}
g_A(\lambda) &= \{\max_d 100d_A - \lambda d_A : 0 \le d_A \le 90\} \\
g_B(\lambda) &= \{\max_d 20d_B - \lambda d_B : 0 \le d_B \le 100\} \\
g_C(\lambda) &= \{\max_p \lambda p_C - 30p_C : p_C = 100u_C, u_C \in \{0, 1\}\} \\
g_D(\lambda) &= \{\max_p \lambda p_D - 90p_D : 0 \le p_D \le 90\}
\end{aligned}$$

True/false (with justification):
The functions $g_A(\lambda), g_B(\lambda), g_C(\lambda), g_D(\lambda)$ can be expressed in closed form as follows:

$$g_A(\lambda) = \begin{cases} 0, & \lambda > 100 \\ 9000 - 90\lambda, & \lambda \le 100 \end{cases}$$
$$g_B(\lambda) = \begin{cases} 0, & \lambda > 20 \\ 2000 - 100\lambda, & \lambda \le 20 \end{cases}$$
$$g_C(\lambda) = \begin{cases} 0, & \lambda \le 30 \\ 100\lambda - 3000, & \lambda \le 30 \end{cases}$$
$$g_D(\lambda) = \begin{cases} 0, & \lambda \le 90 \\ 90\lambda - 8100, & \lambda \le 90 \end{cases}$$

Question 3: Depict $g(\lambda) = g_A(\lambda) + g_B(\lambda) + g_C(\lambda) + g_D(\lambda)$ graphically.

Question 4: sFor which λ is $g(\lambda)$ minimized, and what is the value of $d^\star$? Is there a duality gap, i.e. $d^\star > p^\star$?

Solution

Question 1: The problem is essentially an economic dispatch for two possible values of u_C (equal to 0 or 1). The optimal solution is:

$$u_C = 1$$
$$d_A = 90$$
$$d_B = 10$$
$$p_C = 100$$
$$p_D = 0$$

The objective function is equal to $90 \cdot 100 + 10 \cdot 20 - 100 \cdot 30 = 6200$.

Question 2: True.

- For a value of $\lambda > 100$, we have an optimal consumption for agent A equal to 0. For $\lambda \leq 100$, we have an optimal consumption for agent A equal to 90, thus the profit of agent A (the function $g_A(\lambda)$) is $(100 - \lambda) \cdot 90 = 9000 - 90\lambda$.

- For a value of $\lambda > 20$, we have an optimal consumption for agent B equal to 0. For $\lambda \leq 20$, we have an optimal consumption for agent B equal to 100, thus the profit of agent B (the function $g_B(\lambda)$)) is $(20 - \lambda) \cdot 100 = 2000 - 100\lambda$.

- For a value of $\lambda \leq 30$, we have an optimal production for agent B equal to 0. For $\lambda > 30$, we have an optimal production for agent C equal to 100, thus the profit of agent C (the function $g_C(\lambda)$)) is $(\lambda - 30) \cdot 100 = 100\lambda - 3000$.

- For a value of $\lambda \leq 90$, we have an optimal production for agent D equal to 0. For $\lambda > 90$, we have an optimal production for agent D equal to 90, thus the profit of agent D (the function $g_D(\lambda)$)) is $(\lambda - 90) \cdot 90 = 90\lambda - 8100$.

Question 3: For $\lambda \leq 20$ we have the following sum:

$$(9000 - 90\lambda) + (2000 - 100\lambda) + 0 + 0 = 11000 - 190\lambda$$

For $20 < \lambda \leq 30$:

$$(9000 - 90\lambda) + 0 + 0 + 0 = 9000 - 90\lambda$$

For $30 < \lambda \leq 90$:

$$(9000 - 90\lambda) + 0 + (100\lambda - 3000) + 0 = 6000 + 10\lambda$$

For $90 < \lambda \leq 100$:

$$(9000 - 90\lambda) + 0 + (100\lambda - 3000) + (90\lambda - 8100) = -2100 + 100\lambda$$

For $\lambda > 100$:

$$0 + 0 + (100\lambda - 3000) + (90\lambda - 8100) = -11100 + 190\lambda$$

We know that the function is concave (thus continuous), therefore it suffices to compute the points where the slope changes. In particular:

- At $\lambda = 20$ we have $g(20) = 7200$
- At $\lambda = 30$ we have $g(30) = 6300$
- At $\lambda = 90$ we have $g(90) = 6900$
- At $\lambda = 100$ we have $g(100) = 7900$

The dual function is depicted in figure 2.1.

Question 4: The function is minimized at the point where its slope changes from negative to positive, i.e. for $\lambda = 30$. And we have $d^\star = g(30) = 6300$, which is greater than 6200, therefore there is a duality gap.

Problem 2.3: Consider the following optimization problem:

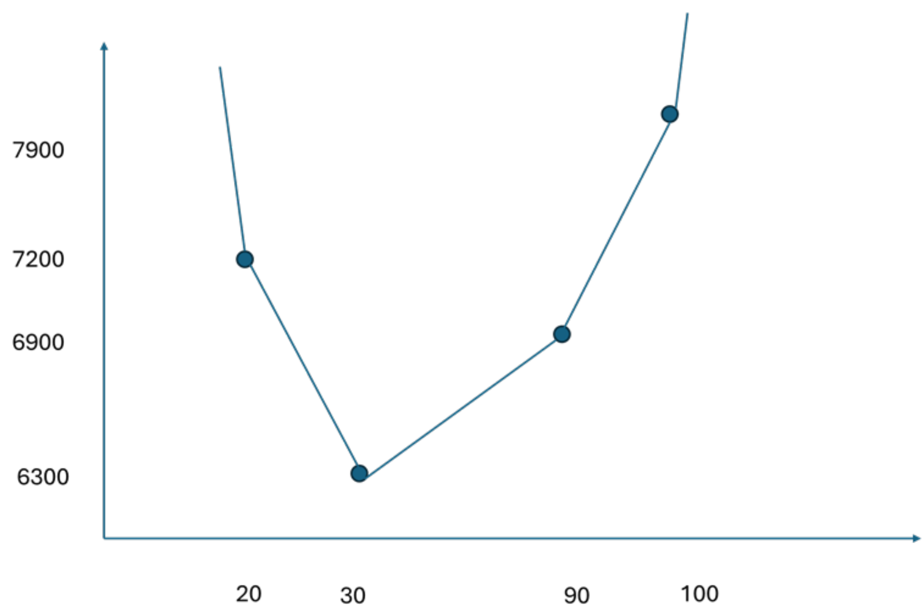
$$\begin{aligned} & \min_x x \\ (\nu): \quad & x^3 = 0 \\ (\lambda): \quad & x \geq 0 \end{aligned}$$

Question 1: Define the dual function as

$$g(\lambda, \nu) = \min_x \{x - \lambda \cdot x - \nu \cdot x^3\}$$

True/false with justification: The problem has a zero duality gap.

Figure 2.1: The dual function of problem 2.2.



Question 2: The KKT conditions of the problem are:

$$0 \leq x \perp \lambda \geq 0, x^3 = 0, 1 - \lambda - \nu \cdot 3 \cdot x^2 = 0$$

True/false with justification: Every point (x, λ, ν) that satisfies the KKT conditions corresponds to an optimal solution of the primal and dual problem.

Solution

Question 1: True. The dual function $g(\lambda, \nu)$ is

$$g(\lambda, \nu) = \begin{cases} 0, & \lambda = 1, \nu = 0 \\ -\infty, & \text{otherwise} \end{cases}$$

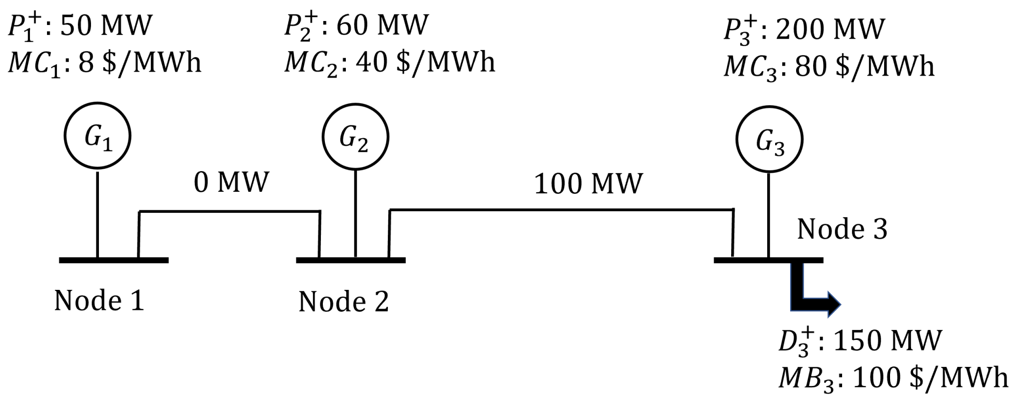
The dual function maximum is $d^\star = 0$. The optimal objective function value of the primal problem is $p^\star = 0$.

Question 2: False. Let us consider the KKT point $x = 0, \lambda = 1, \nu = 1$. This point is not optimal for the dual problem.

This specific example establishes that we can have a zero duality gap, but with the KKT conditions not being sufficient for characterizing an optimal solution.

Problem 2.4: Consider the following optimal power flow problem, which corresponds to the system in figure 2.2.

Figure 2.2: The system of problem 2.3.



$$
\begin{aligned}
p^\star \quad &= \max_{p\geq 0, d\geq 0, r, f} 100\cdot d_3 - 8\cdot p_1 - 40\cdot p_2 - 80\cdot p_3 \\
(\lambda_1): \quad & r_1 - p_1 = 0 && (2.1)\\
(\lambda_2): \quad & r_2 - p_2 = 0 && (2.2)\\
(\lambda_3): \quad & r_3 + d_3 - p_3 = 0 && (2.3)\\
& p_1 \leq 50 && (2.4)\\
& p_2 \leq 60 && (2.5)\\
& p_3 \leq 200 && (2.6)\\
& d_3 \leq 150 && (2.7)\\
& f_{1-2} = r_1 && (2.8)\\
& f_{2-3} = r_1 + r_2 && (2.9)\\
& f_{1-2} \leq 0 && (2.10)\\
& f_{1-2} \geq 0 && (2.11)\\
& f_{2-3} \leq 100 && (2.12)\\
& f_{2-3} \geq -100 && (2.13)\\
& r_1 + r_2 + r_3 = 0 && (2.14)
\end{aligned}
$$

The problem corresponds to an optimal power flow: we aim at satisfying loads without violating the technical limits of power lines. Our goal is to maximize social welfare, which is the difference between the benefit of consumers who use electricity and the cost of producers for generating the electricity.

The variable r_i corresponds to the amount of electricity which is injected in node i of the network and is withdrawn in the reference node (node 3 in our problem) and the variable f_{i-j} corresponds to the flow of power from node i to node j of the network.

Question 1: Match the following constraints to their corresponding numbering:

- C1: conservation of energy. The total power injected into the system equals the total power withdrawn from the system.

- C2: consumer 3 will consume more than 150 MW of power.

- C3: generator 2 cannot produce more than 60 MW.

- C4: The flow of power from node 3 to node 2 cannot exceed 100 MW.

- C5: The flow of power on line 2-3 equals the injection of power in node 1 and the injection of node in node 2.

- C6: The injection of power in node 3 equals the production of power in node 3 minus the consumption in node 3.

Question 2: We define the following dual function:

$$\begin{aligned} S(\lambda_1, \lambda_2, \lambda_3) \quad = \quad & \max_{p \geq 0, d \geq 0, r, f} 100 \cdot d_3 - 8 \cdot p_1 - 40 \cdot p_2 - 80 \cdot p_3 \\ & -\lambda_1 \cdot (r_1 - p_1) - \lambda_2 \cdot (r_2 - p_2) - \lambda_3 \cdot (r_3 + d_3 - p_3) \\ & p_1 \leq 50 \\ & p_2 \leq 60 \\ & p_3 \leq 200 \\ & d_3 \leq 150 \\ & f_{1-2} = r_1 \\ & f_{2-3} = r_1 + r_2 \\ & f_{1-2} \leq 0 \\ & f_{1-2} \geq 0 \\ & f_{2-3} \leq 100 \\ & f_{2-3} \geq -100 \\ & r_1 + r_2 + r_3 = 0 \end{aligned}$$

Question 2: True/false with justification: It is true that $S(\lambda_1, \lambda_2, \lambda_3) \geq p^\star$ for any $(\lambda_1, \lambda_2, \lambda_3)$.

Question 3: True/false with justification: The function $S(\lambda_1, \lambda_2, \lambda_3)$ is concave.

Question 4: Compute $S(40, 40, 80)$. Exploit the observation that the function $S(\lambda_1, \lambda_2, \lambda_3)$ decomposes into three types of sub-problems for market prices $(\lambda_1, \lambda_2, \lambda_3)$: (i) one profit maximization sub-problem for each producer, (ii) one surplus maximization sub-problem for each consumer, (iii) one surplus maximization sub-problem for the system operator who buys power in cheap locations and sells it in expensive locations.

Solution

Question 1:
We have the following correspondence:

- C1 corresponds to constraint (2.14)
- C2 corresponds to constraint (2.7)

- C3 corresponds to constraint (2.5)
- C4 corresponds to constraint (2.13)
- C5 corresponds to constraint (2.9)
- C6 corresponds to constraint (32.3)

Question 2: True. The function $S(\lambda_1, \lambda_2, \lambda_3)$ is the relaxation of a maximization problem, thus it is an upper bound to the objective function of the primal problem $p^\star$.

Question 3: False. The function $S(\lambda_1, \lambda_2, \lambda_3)$ is the dual function of a maximization problem, and is therefore convex.

Question 4: We write the dual function as:

$$S(\lambda_1, \lambda_2, \lambda_3) = S_{G_1}(\lambda_1) + S_{G_2}(\lambda_2) + S_{G_3}(\lambda_3) + S_{L_3}(\lambda_3) + S_N(\lambda_1, \lambda_2, \lambda_3)$$

where

$$\begin{aligned} S_{G_1}(\lambda_1) &= \max_{p_1 \geq 0} (\lambda_1 - 8) \cdot p_1 \\ & \quad p_1 \leq 50 \end{aligned}$$

The profit for $\lambda_1 = 40$ with simple inspection is $32 \cdot 50 = 1600$ \$ since the optimal production is 50 MW.

$$\begin{aligned} S_{G_2}(\lambda_2) &= \max_{p_2 \geq 0} (\lambda_2 - 40) \cdot p_2 \\ & \quad p_2 \leq 60 \end{aligned}$$

The profit for $\lambda_2 = 40$ with simple inspection is 0 and any feasible production is optimal.

$$\begin{aligned} S_{G_3}(\lambda_3) &= \max_{p_3 \geq 0} (\lambda_3 - 80) \cdot p_3 \\ & \quad p_3 \leq 200 \end{aligned}$$

The profit for $\lambda_3 = 80$ with simple inspection is 0 and any feasible production is optimal.

$$\begin{aligned} S_{L_3}(\lambda_3) &= \max_{d_3 \geq 0} (100 - \lambda_3) \cdot d_3 \\ & d_3 \leq 150 \end{aligned}$$

The surplus for $\lambda_3 = 80$ with simple inspection is $20 \cdot 150 = 3000$ \$ since the optimal consumption is equal to 150 MW.

$$\begin{aligned} & S_N(\lambda_1, \lambda_2, \lambda_3) = \max_{r,f} -\lambda_1 \cdot r_1 - \lambda_2 \cdot r_2 - \lambda_3 \cdot r_3 \\ & f_{1-2} = r_1 \\ & f_{2-3} = r_1 + r_2 \\ & r_1 + r_2 + r_3 = 0 \\ & f_{1-2} \leq 0 \\ & f_{1-2} \geq 0 \\ & f_{2-3} \leq 100 \\ & f_{2-3} \geq -100 \\ & r_1 + r_2 + r_3 = 0 \end{aligned}$$

Since $f_{1-2} = 0$, this implies that $r_1 = 0$.s Thus the above problem is equivalent to

$$\begin{aligned} & \min_{r,f} \lambda_2 \cdot r_2 + \lambda_3 \cdot r_3 \\ & f_{2-3} = r_2 \\ & r_2 + r_3 = 0 \\ & f_{2-3} \leq 100 \\ & f_{2-3} \geq -100 \\ & r_2 + r_3 = 0 \end{aligned}$$

Substituting out r_2, we conclude that the above is equivalent to

$$\begin{aligned} & \max_{r,f} -(\lambda_3 - \lambda_2) \cdot r_3 \\ & f_{2-3} = -r_3 \\ & f_{2-3} \leq 100 \\ & f_{2-3} \geq -100 \end{aligned}$$

Which is finally equivalent to

$$\max_{r,f}(\lambda_3 - \lambda_2) \cdot f_{2-3}$$
$$f_{2-3} \leq 100$$
$$f_{2-3} \geq -100$$

Thus the agent is paid to transfer power from node 2 to node 3. Itts surplus for $\lambda_2 = 40$ and $\lambda_3 = 80$ is $40 \cdot 100 = 4000$ $ since through simple inspection the optimal solution is to transfer 100 MW from node 2 to node 3.
We thus have that $S(40, 40, 80) = 1600 + 0 + 0 + 3000 + 4000 = 8600$.

Problem 2.5: Solve the following optimization problem:

$$\max_{0 \leq \lambda \leq 1} \lambda^2 + (1 - \lambda)^2$$

Is it a convex optimization problem?

Solution: The problem is not convex, because we are maximizing a convx function. The maximum is achieved at $\lambda = 0$ or $\lambda = 1$, and is equal to 1.

Problem 2.6: T/F with justification: Every point on the boundary of the disk $\{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ (i.e. every point x with $x_1^2 + x_2^2 = 1$) is an extreme point of the disk. Recall that an extreme point of a set is a point which cannot be expressed as the convex combination of two other different points that belong to the set.
Hint: You can use the triangle inequality ($\|w + y\|^2 \leq \|w\|^2 + \|y\|^2$) and the result of problem 2.5.

Solution: True: Any point on the boundary of the disk is an extreme point. In order to see why this is the case, let us prove it by contradiction. Concretely, let us assume that there is a point on the boundary of the disk, x, which is expressed as a convex combination of the points x_1 and x_2. We then have

$$x = \lambda x_1 + (1 - \lambda)x_2$$

for some $0 < \lambda < 1$. Therefore

$$\|x\|^2 = \|\lambda x_1 + (1 - \lambda)x_2\|^2$$
$$\leq \lambda^2\|x_1\|^2 + (1 - \lambda)^2\|x_2\|^2$$
$$\leq \lambda^2 + (1 - \lambda)^2$$

where the first inequality applies the triangular inequality (Cauchy-Schwarz inequality) and the second inequality is a consequence of the fact that x_1 and x_2 belong to the disk.
This expression is maximized for $\lambda = 0$ or $\lambda = 1$ in the interval $0 \leq \lambda \leq 1$, any other value of λ in this interval leads to a result less than 1, see problem 2.5. It therefore follows that

$$\|x\|^2 < 1$$

for any $0 < \lambda < 1$. Which is a contradiction because we assumed that x belongs to the boundary of the disk, thus $\|x\|^2 = 1$.

Problem 2.7: In your weekly meeting, your PhD student reports that the same optimization problem gives two different solutions, but with the same objective function value, when running the problem in two different machines. Can this be true?
In the same meeting, your student reports that he/she obtains a different objective function value for the same optimization problem when running the problem in two different machines. Can this be true?
Solution: Regarding the firs question, the answer is yes. For instance, $\min_x 0$ has infinitely many optimal solutions.
Regarding the second question, the answer is no. By definition, one of the solutions is sub-optimal.

Chapter 3

Power system operations and power market operations

Problem 3.1 Suppose that the market consists of 1000 MW of natural gas, 4000 MW of coal, and 1000 MW of renewable energy sources. The marginal cost of renewable energy sources is 0 \$/MWh and the marginal cost of coal is 20 \$/MWh. The marginal cost of natural gas is 100 \$/MWh in January 2024, but due to some geopolitical disruption it rockets to 450 \$/MWh in February 2024. The inverse demand function of the market for both months is

$$MB(Q) = 2.56 \cdot 10^6 \text{ \$/MWh}.$$

Assuming a perfectly competitive market, compute the market price in January and February 2024.

Solution: We can graphically compute that the market clearing price is set by natural gas, and is equal to 100 \$/MWh in January 2024 and 450 \$/MWh in February 2024. For the sake of confirmation, demand at 100 \$/MWh is 5119.8 MW, which indeed exceeds 5000 MW and is less than 6000 MW thus natural gas is the marginal technology, whereas at 450 \$/MWh the demand is 5119.1 MW, which is also between 5000 and 6000 MW.

Problem 3.2 The green retail tariff that was proposed by the Greek government in 2023 is announced at the beginning of each month, and is defined according to the following formula:
Retail charge = (final retail price) × (monthly consumption [kWh]) + fixed retail charge

Table 3.1: The parameters used in exercise 3.2.

Supplier	Fixed retail price [$/month]	Base retail price [$/kWh]	α	L_U [$/kWh]	L_L [$/kWh]
S1	5	0.128	1.20	0.095	0.085
S2	5	0.090	1.30	0.055	0.045

Here,
Final retail price = base retail price + variation mechanism
The variation mechanism for month M is defined as follows:

$$\text{Variation mechanism} = \begin{cases} \alpha \cdot (TEA_{M-1} - L_L) + \beta, & TEA_{M-1} < L \\ 0, & L_L \leq TEA_{M-1} \leq L_U \\ \alpha \cdot (TEA_{M-1} - L_U), & TEA_{M-1} > L_U \end{cases}$$

where

$$\beta = \alpha \cdot (TEA_{M-1} - TEA_{M-2}) \text{ [\$/kWh]}$$

and TEA_M (in $/kWh) is the average market price during month M, with parameters α, L_L and L_U being decided by retail suppliers.

1. Let us consider the marekt clearing prices of problem 3.1. What is the final retail price for March 2024 given the parameters that are presented in table 3.1? What is the retail charge under each supplier for a consumer who consumes 450 kWh in March?

2. Supplier S2 mentions in TV spots that it offers the lowest base retail price. Would you select supplier S2? Can different suppliers have a different retail price under the green tariff? Is it necessary for consumers to know the exact mathematical formula by which the final retail price is computed in order to select a retail supplier for March?

Question 1: We compute the following for the two suppliers:

$$\begin{aligned} \beta_1 &= \alpha_1 \cdot (TEA_{M-1} - TEA_{M-2}) = \\ &\quad 1.2 \cdot (450 - 100) = 420 \text{ \$/MWh} = 0.42 \text{ \$/kWh} \\ \beta_2 &= \alpha_2 \cdot (TEA_{M-1} - TEA_{M-2}) = \\ &\quad 1.3 \cdot (450 - 100) = 455 \text{ \$/MWh} = 0.455 \text{ \$/kWh} \end{aligned}$$

The variation mechanism for the two suppliers is computed as follows:

$$\begin{aligned}
VM_1 &= \alpha_1 \cdot (TEA_{M-1} - L_{U,1}) + \beta_1 = \\
&\quad 1.2 \cdot (0.450 - 0.095) + 0.042 = 0.846 \text{ \$/kWh} \\
VM_2 &= \alpha_2 \cdot (TEA_{M-1} - L_{U,2}) + \beta_2 = \\
&\quad 1.3 \cdot (0.450 - 0.055) + 0.455 = 0.9685 \text{ \$/kWh}
\end{aligned}$$

The final retail price for the two suppliers is:

$$FRP_1 = BRP_1 + VM_1 = 0.128 + 0.846 = 0.974 \text{ \$/kWh}$$
$$FRP_2 = BRP_2 + VM_2 = 0.090 + 0.9685 = 1.0585 \text{ \$/kWh}$$

The retail charge for a consumption of 450 kWh under each of the two suppliers is:

$$RC_1 = 0.974 \cdot 450 + 5 = \$443.3$$
$$RC_2 = 1.0585 \cdot 450 + 5 = \$481.3$$

Question 2: Supplier S2 is not preferable. Although it offers the lowest basic retail price, it offers less favorable parameters for computing the final retail price.
We can have different prices across suppliers, even if the mathematical formula is the same, because different suppliers can use different parameter values for the formula.
It is not necessary for consumers to know the price math formula in order to select a supplier, only the final price.

Problem 3.3 A system consists of one coal unit and one natural gas unit. The marginal cost functions of the two technologies are as follows:

$$MC_{NG}(p) = \begin{cases} 0.5p \text{ \$/MWh}, & 0 \leq p \leq 100 \text{ MW} \\ 70 \text{ \$/MWh}, & 100 < p \leq 120 \text{ MW} \end{cases}$$
$$MC_C(p) = 0.125p \text{ \$/MWh}, 0 \leq p \leq 100 \text{ MW}$$

1. Compute the aggregate marginal cost curve $MC(p)$ of the system.
2. Depict the aggregate marginal cost curve of the system $MC(p)$ graphically.

3. What is the marginal system price (the system lambda) for a load of 180 MW?

Solution:
Question 1: We split the production of natural gas into two parts, $p_{NG,1}$ and $p_{NG,2}$. We consider the following cases:

- For $p < 125$: $p_{NG,1} > 0$ and $p_C > 0$ (we know that production goes up to 125 MW because in this case coal produces 4 times more than natural gas)
- For $125 \leq p \leq 200$: $p_C = 100$, $p_{NG,1} < 100$
- For $200 < p < 220$: $p_C = 100$, $p_{NG,1} = 100$, $p_{NG,2} > 0$

In case 1 the marginal cost of natural gas and lignite are equal, and we specifically have that

$$0.5p_{NG,1} = 0.125p_C \Rightarrow p_C = 4p_{NG,1}$$
$$p_{NG,1} + p_C = p \Rightarrow p_{NG,1} = 0.2p, p_C = 0.8p$$

Thus in this case we have that the aggregate marginal cost is equal to the marginal cost of each technology:

$$MC(p) = MC_{NG}(p_{NG}) = 0.5 \cdot 0.2p = 0.1p$$

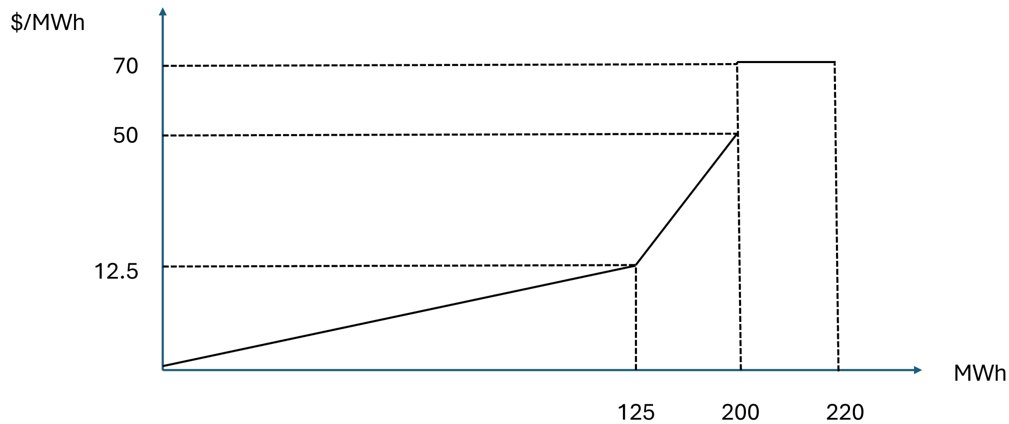
But also:

$$MC(p) = MC_C(p_C) = 0.125 \cdot 0.8p = 0.1p$$

Thus, in case 1 we have $MC(p) = 0.1p$.
In case 2 we have $MC(p) = 0.5 \cdot (p - 100)$ (the marginal cost of natural gas, which is producing $p - 100$ in case 2)
In case 3 we have $MC(p) = 70$.
Question 2: For the graphical representation we require the marginal cost at 125 MW and 200 MWh:

- In case 1 at 125 MWh: 12.5 \$/MWh
- In case 2 at 125 MWh: 12.5 \$/MWh
- In case 2 at 200 MWh: 50 \$/MWh

Figure 3.1: The aggregate marginal cost curve of problem 3.3.



The aggregate marginal cost curve is presented in figure 3.1.
Question 3: We find ourselves in case 2, thus we have that $MC(180) = 0.5 \cdot (180 - 100) = 40$ \$/MWh.

Problem 3.4

Question 1: Consider a marginal cost curve given by the following relation:

$$MC(Q) = 10 + Q.$$

What is the elasticity for $Q = 0$? What is the elasticity for $Q \to \infty$?

Question 2: Consider a marginal cost curve given by the following relation:

$$MB(Q) = 1000 - 0.1 \cdot Q.$$

What is the elasticity for $Q = 0$? What is the elasticity for $Q = 10000$? For which Q does the elasticity become equal to -1?

Solution

Question 1: For $Q = 0$ the elasticity is equal to

$$\epsilon = \frac{dQ/Q}{dP/P} = \frac{dQ}{dP}\frac{P}{Q} = 1\frac{10}{0} = +\infty$$

For $Q \to \infty$ the elasticity is given as

$$\epsilon = \frac{dQ}{dP}\frac{P}{Q} = 1 \cdot \lim_{Q\to\infty} \frac{10+Q}{Q} = 1$$

Question 2: For $Q = 0$ h elasticity is given as

$$\epsilon = \frac{dQ}{dP}\frac{P}{Q} = -10 \cdot \frac{10000}{0} = -\infty$$

For $Q = 10000$ the elasticity is given as

$$\epsilon = \frac{dQ}{dP}\frac{P}{Q} = -10 \cdot \frac{10000 - 0.1 \cdot Q}{10000} = 0$$

The elasticity becomes -1 when

$$\epsilon = \frac{dQ}{dP}\frac{P}{Q} = -10 \cdot (10000 - 0.1 \cdot Q) \Rightarrow 0.1 \cdot Q = 10000 - 0.1 \cdot Q \Rightarrow Q = \frac{1000}{0.2} = 5000$$

Chapter 4

Economic dispatch

Problem 4.1: A uniform price auction consists of the following offers:

- Offer B1: Buy 70 MWh at 100 $/MWh
- Offer B2: Buy 30 MWh at 300 $/MWh
- Offer B3: Buy 25 MWh at 85 $/MWh
- Offer S1: Sell 80 MWh at 40 $/MWh
- Offer S2: Sell 20 MWh at 10 $/MWh
- Offer S3: Sell 35 MWh at 120 $/MWh

1. Write out the economic dispatch problem which corresponds to these offers, as well as the KKT conditions of the problem.
2. What is the optimal matching of orders?
3. Is there a unique market clearing price? If yes, use the KKT conditions of the problem in order to prove that this price clears the market. If there are multiple clearing prices, what is their range, and use the KKT conditions in order to show that the minimum and maximum price in this range actually clear the market.

Solution:

1. The economic dispatch problem can be expressed as follows:

$$
\begin{aligned}
& \max_{p,d} 100d_1 + 300d_2 + 85d_3 - 40p_1 - 10p_2 - 120p_3 \\
(\nu_1): \quad & d_1 \leq 70 \\
(\nu_2): \quad & d_2 \leq 30 \\
(\nu_3): \quad & d_3 \leq 25 \\
(\mu_1): \quad & p_1 \leq 80 \\
(\mu_2): \quad & p_2 \leq 20 \\
(\mu_3): \quad & p_3 \leq 35 \\
(\lambda): \quad & d_1 + d_2 + d_3 - p_1 - p_2 - p_3 = 0 \\
& p \geq 0, d \geq 0
\end{aligned}
$$

   The KKT conditions are as follows:

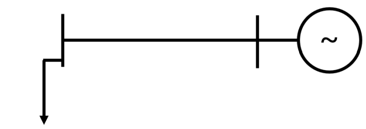
$$
\begin{aligned}
& d_1 + d_2 + d_3 - p_1 - p_2 - p_3 = 0 \\
& 0 \leq \nu_1 \perp 70 - d_1 \geq 0 \\
& 0 \leq \nu_2 \perp 30 - d_2 \geq 0 \\
& 0 \leq \nu_3 \perp 25 - d_3 \geq 0 \\
& 0 \leq \mu_1 \perp 80 - p_1 \geq 0 \\
& 0 \leq \mu_2 \perp 20 - p_2 \geq 0 \\
& 0 \leq \mu_3 \perp 35 - p_3 \geq 0 \\
& 0 \leq p_1 \perp 40 + \mu_1 - \lambda \geq 0 \\
& 0 \leq p_2 \perp 10 + \mu_2 - \lambda \geq 0 \\
& 0 \leq p_3 \perp 120 + \mu_3 - \lambda \geq 0 \\
& 0 \leq d_1 \perp -100 - \nu_1 + \lambda \geq 0 \\
& 0 \leq d_2 \perp -300 - \nu_2 + \lambda \geq 0 \\
& 0 \leq d_3 \perp -85 - \nu_3 + \lambda \geq 0
\end{aligned}
$$

2. The optimal matching of orders is as follows:

   - $d_1 = 70$, $d_2 = 30$, $d_3 = 0$
   - $p_1 = 80$, $p_2 = 20$, $p_3 = 0$

3. There are multiple market clearing prices, ranging from 85 to 100 \$/MWh. We need to propose the price, but also the profit margin,

Figure 4.1: The system of problem 4.2.



and plug them into the KKT conditions, in order to verify that the primal-dual solution is indeed optimal. For instance, if we propose $\lambda = 85$ \$/MWh, then we have $\nu_1 = 15$, $\nu_2 = 215$, $\nu_3 = 0$, and $\mu_1 = 45$, $\mu_2 = 75$, $\mu_3 = 0$.

Problem 4.2: Consider the system corresponding to figure 4.1.
The market clearing problem is expressed as follows:

$$
\begin{aligned}
& \min_{p,f} 10 \cdot p \\
(\rho_1): \quad & 35 + f = 0 \\
(\rho_2): \quad & -f - p = 0 \\
(\lambda^+): \quad & f \leq 100 \\
(\lambda^-): \quad & -f \leq 100 \\
(\mu): \quad & p \leq 50 \\
& p \geq 0
\end{aligned}
$$

We will apply proposition 4.11 of the textbook to the special case of this problem.

Question 1: Express the KKT conditions of the problem.

Question 2: Match the KKT conditions to the following four categories:

- Price adjustment of the energy market in node 1

- Price adjustment of the energy market in node 2
- Profit maximization of producer
- Surplus maximization of the transmission system operator

Question 3: Express the surplus maximization problem of the transmission system operator. Which are the private variables of the rusplus maximization problem of the system operator? Which are the private constraints? What does the objective function express?

Question 4: Compute the primal-dual optimal pair.

Question 5: What is the surplus of the transmission system operator (the optimal value of the objective function of question 3) at the optimal solution?

Solution

Question 1: The KKT conditions are expressed as follows:

$$
\begin{aligned}
&35 + f = 0 \\
&-f - p = 0 \\
&0 \leq \lambda^+ \perp 100 - f \geq 0 \\
&0 \leq \lambda^- \perp f + 100 \geq 0 \\
&0 \leq \mu \perp 50 - p \geq 0 \\
&0 \leq p \perp -\rho_2 + 10 + \mu \geq 0 \\
&\rho_1 - \rho_2 + \lambda^+ - \lambda^- = 0
\end{aligned}
$$

Question 2: The matching is as follows:

- Price adjustment at node 1 corresponds to the first condition.
- Price adjustment at node 2 corresponds to the second condition.
- Producer profit maximization corresponds to the fifth and sixth condition.

- Surplus maximization of the transmission system operator corresponds to the third, fourth and seventh condition.

Question 3: The surplus maximization problem is expressed as follows:

$$\max_{f} \rho_2 \cdot f - \rho_1 \cdot f$$
$$(\lambda^+): \quad f \leq 100$$
$$(\lambda^-): \quad -f \leq 100$$

The private decision variable of the system operator is the flow on the link, f. The private constraints are the upper and lower bound on the line flow. The private objective function describes the fact that the system operator is paid for buying energy at node 1 at the local price and sells this energy at node 2.

Question 4: The generator produces $p = 35$ and the flow is $f = -35$. We have $\lambda^+ = \lambda^- = 0$ and $\mu = 0$. We have $\rho_2 = 10$ and $\rho_1 = 10$.

Question 5: The value is 0 $.

Problem 4.3: You are given the data of the folder "ProblemAE43" regarding the Greek market in 2024. The dates in question are July 3 and July 22, 2024, which were respectively a relatively calm and a difficult day during the summer of 2024, with moderate and very high electricity prices respectively.

Question 1: Implement a perfect competition model with the following characteristics:

- Elastic marginal value curve in the form $\lambda(Q) = A - B \cdot Q$, where A and B are given in the flies AlphaD.txt and BetaD.txt respectively, and a set of loads given in Loads.txt
- Representation of all thermal units in the Greek system (set of generators in Generators.txt, marginal costs in MargCost.txt, technical maxima in PMax.txt)
- Imports, domestic renewable supply and hydro production in Offset.txt, where positive values in the file correspond to production and negative ones to consumption

Question 2: Implement a perfect competition model which has the following additional characteristics relative to the model of question 1:

- Endogenous management of hydro, where the only thing you know is how much hydro was produced throughout the day, the information is available in TotalHydro.txt

- You need to subtract the hydro from question 1, thus you are given a file named OffsetWithoutHydro.txt which yuo need to use for the model of question 2

Question 3: Find the market clearing prices for these dates, and compare them to the prices predicted by your two models. Are your predictions accurate? Why do you believe that there are possible differences?

Question 4: Compute the profitability of thermal units that is predicted by your model in questions 1 and 2. Compare to the profitability that is achieved by producers from thermal units if they are facing the prices as they occurred historically. Comment on your results.

Solution

Question 1: The code is available in the textbook website. You need to comment the lines indicated in the dat file when running question 1. The date is chosen by commenting the appropriate lines in the mod file.
The prices for July 3 are as follows: 1 82.2343 2 82.2343 3 82.4806 4 84.1985 5 84.1985 6 85.6403 7 84.1985 8 83.6095 9 84.1985 10 82.4806 11 82.4806 12 82.4806 13 82.4806 14 82.4806 15 83.6095 16 84.1985 17 84.1985 18 84.1985 19 84.1985 20 83.6095 21 83.6095 22 82.4806 23 83.6095 24 82.4806
The prices for July 22 are as follows: 1 84.1985 2 84.6601 3 85.6403 4 88.5139 5 88.6124 6 88.6124 7 88.5139 8 88.5139 9 88.5139 10 88.5139 11 85.6403 12 84.1985 13 84.1985 14 85.6403 15 85.6403 16 85.6403 17 88.6124 18 95.8911 19 88.6124 20 88.5139 21 85.6403 22 85.6403 23 84.1985 24 84.1985

Question 2: The code is available in the textbook website. You need to comment the lines indicated in the dat file when running question 2. The date is chosen by commenting the appropriate lines in the mod file.

The prices for July 3 are as follows: 1 82.2343 2 82.2343 3 83.6095 4 83.6095 5 83.6095 6 83.6095 7 83.6095 8 83.6095 9 83.6095 10 82.4806 11 82.4806 12 82.4806 13 82.4806 14 83.6095 15 83.6095 16 83.6095 17 83.6095 18 83.6095 19 83.6095 20 83.6095 21 83.6095 22 83.6095 23 83.6095 24 83.3029
The prices for July 22 are as follows: 1 85.6403 2 85.6403 3 88.5139 4 88.5139 5 88.5139 6 88.5139 7 88.5139 8 88.5139 9 88.5139 10 88.5139 11 88.5139 12 85.6403 13 88.5139 14 88.5139 15 88.5139 16 88.5139 17 88.5139 18 88.5139 19 88.5139 20 88.5139 21 88.5139 22 88.5139 23 85.6403 24 85.6403

Question 3: The prices for July 3, as recorded in the power exchange (and available in the excel file titled da_prices), are: 1 93.79 2 95 3 99.05 4 110.43 5 121.29 6 128.43 7 99.23 8 99.61 9 99.04 10 97.92 11 97.95 12 99.05 13 124 14 130 15 115.03 16 129.22 17 160.09 18 212.57 19 168.8 20 127.82 21 107.62 22 109.3 23 99.04 24 99.05
The prices for July 22, as recorded in the power exchange (and available in the excel file titled da_prices), are: 1 100 2 100 3 104 4 154.15 5 161.14 6 134.47 7 131 8 112.61 9 108.04 10 112.86 11 129.41 12 131.19 13 154.15 14 185.15 15 201.37 16 384.58 17 751.1 18 850 19 666.54 20 150 21 121.6 22 120.83 23 105.1 24 102
There are clear deviations, which can be attributed to a number of factors, including but not limited to the following:

- Our model does not account for the exercise of market power.
- Our model does not represent reserves.
- Our model does not account for fixed costs.
- Our model does not account for possible availability below nominal capacity.

Question 4: No code is available for this question for the moment.

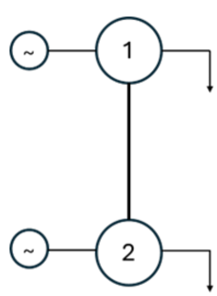
Chapter 5

The transmission network

Problem 5.1 The system in figure 5.1 consists of two nodes with generators and loads at both nodes, and a line with a capacity limit of 100 MW connecting the two nodes.
Compute the locational marginal prices, the production and the consumption at each node, the power injection at each node, and the power flow on the line at the optimal power flow solution, for the following data (where Q is measured in MW):

- Marginal cost of producers in node 1: $MC_1(Q) = 10 + 0.05Q$ \$/MWh
- Marginal cost of producers in node 2: $MC_2(Q) = 42 + 0.025Q$ \$/MWh
- Marginal benefit of consumers in node 1: $MB_1(Q) = 37.5 - 0.05Q$ \$/MWh

Figure 5.1: The network of problem 5.1.



- Marginal benefit of consumers in node 2: $MB_2(Q) = 80 - 0.1Q$ \$/MWh

Compute the following payments:

- From the system operator to the producers of node 1
- From the system operator to the producers of node 2
- From the consumers of node 1 to the system operator
- From the consumers of node 2 to the system operator
- The congestion rent of the system operator

Solution: Let us assume that the line is congested. The the following conditions must hold:

$$\begin{aligned}
&10 + 0.05p_1 = \rho_1 \\
&37.5 - 0.05d_1 = \rho_1 \\
&p_1 - d_1 = 0 \\
&42.5 + 0.025p_2 = \rho_2 \\
&80 - 0.1d_2 = \rho_2 \\
&p_2 - d_2 = 0
\end{aligned}$$

The first subsystem is independent of the second. We thus have:

$$10 + 0.05 \cdot (d_1 + 100) = 37.5 - 0.05d_1$$

From which it follows that:

$$0.1d_1 = 22.5$$

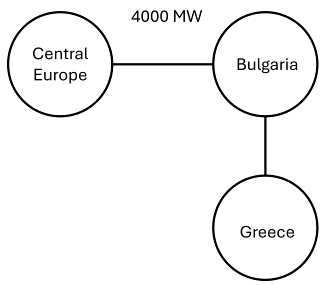
And thus $d_1 = 225$ MW. From which we conclude that $\rho_1 = 26.25$ \$/MWh and $p_1 = 325$ MW. And we confirm that indeed the marginal cost curve at 325 MW is 26.25 \$/MWh.
Similarly, for the second system we have:

$$42 + 0.025p_2 = 80 - 0.1 \cdot (p_2 + 100)$$

From which it follows that:

$$0.125p_2 = 28$$

Figure 5.2: The network of problem 5.2.



And thus $p_2 = 224$ MW. From which we conclude that $\rho_2 = 47.6$ \$/MWh and $d_2 = 324$ MW. And we confirm that indeed the marginal benefit curve at 324 MW is 47.6 \$/MWh.
The injection in node 1 is 100 MW, in node 2 it is -100 MW, and the flow on line 1-2 is 100 MW.
The payments are as follows:

- From the system operator to the producers of node 1: (325 MWh) · (26.25 \$/MWh) = \$8531.25
- From the system operator to the producers of node 2: (224 MWh) · (47.6 \$/MWh) = \$10662.4
- From the consumers of node 1 to the system operator: (225 MWh) · (26.25 \$/MWh) = \$5906.25
- From the consumers of node 2 to the system operator: (324 MWh) · (47.6 \$/MWh) = \$15422.4
- The congestion rent of the system operator: (100 MWh) · (47.6 – 26.25 \$/MWh) = \$2135

Problem 5.2 During the summer of 2024, Greece faced high electricity prices. The system in figure 5.2 is a simplified representation of the phenomenon. We assume that the interconnection between central Europe and Bulgaria has a finite capacity that is presented in figure 5.2.

We assume that central Europe is represented by the following marginal cost curve:

$$MC_{CE}(p) = 20 + 0.045p \text{ \$/MWh.}$$

Bulgaria has a nuclear unit with a zero marginal cost and a capacity of 2000 MW (Kozloduy). The marginal cost curve of Greece is:

$$MC_G(p) = 40 + 0.045p \text{ \$/MWh.}$$

1. Let us assume that the load in Greece in winter is 5000 MW. Compute the equilibrium price at every node of the figure.

2. Let us assume that the load in Greece in the summer is 8000 MW. Compute the equilibrium price at every node.

3. Let us assume that the load in Greece in the summer is 8000 MW and the unit in Kozloduy fails. Compute the equilibrium price at every node of the figure.

4. What would be the price in Greece in question 3 if central Europe were interconnected with limitless line capacity to eastern Europe (Bulgaria and Greece)?

5. What would be the price that Greece faces if in question 3 Greece were interconnected from the European power exchange?

6. At the family table an all-knowing relative explains that "markets are to blame for the high electricity prices in Greece". Based on the previous analysis, can you offer at least three alternative explanations for the high energy prices, and one argument why it may be the case that the exact opposite of what the all-knowing relative supports could be the case?

Solution

Question 1: We assume that the line is not congested, and suppose that the Greek system is producing power. It must be the case that the following conditions hold at equilibrium:

$$p_{CE} + p_G + 2000 = 5000$$
$$20 + 0.045p_{CE} = 40 + 0.045p_G$$

From which we conclude that:

$$\begin{aligned}
& p_G = 3000 - p_{CE} \Rightarrow 20 + 0.045 p_{CE} = 40 + 0.045 \cdot (3000 - p_{CE}) \Rightarrow 0.09 p_{CE} = 38 \\
& \Rightarrow p_{CE} = 1722 \text{ MW}, p_G = 3000 - 1722 = 1278 \text{ MW} \\
& \Rightarrow \lambda_G = \lambda_{CE} = \lambda_{BU} = 20 + 0.045 \cdot 1722 = 40 + 0.045 \cdot 1278 = 97.5 \text{ \$/MWh}
\end{aligned}$$

We further confirm that the line from central Europe to Bulgaria is not congested.

Question 2: We assume again that the line is not congested, and suppose again that the Greek system produces. The following conditions must hold at equilibrium:

$$\begin{aligned}
& p_{CE} + p_G = 8000 \\
& 20 + 0.045 p_{CE} = 40 + 0.045 p_G
\end{aligned}$$

From which we conclude that:

$$\begin{aligned}
& p_G = 6000 - p_{CE} \Rightarrow 20 + 0.045 p_{CE} = 40 + 0.045 \cdot (6000 - p_{CE}) \Rightarrow 0.09 p_{CE} = 290 \\
& \Rightarrow p_{CE} = 3222 \text{ MW}, p_G = 6000 - 3222 = 2778 \text{ MW} \\
& \Rightarrow \lambda_G = \lambda_{CE} = \lambda_{BU} = 20 + 0.045 \cdot 3222 = 40 + 0.045 \cdot 2778 = 165 \text{ \$/MWh}
\end{aligned}$$

We further confirm that the line from central Europe to Bulgaria is not congested.

Question 3: We assume that the line is congested, and suppose again that the Greek system produces. The following conditions must hold at equilibrium:

$$\begin{aligned}
& p_{CE} = 4000 \\
& p_G = 4000 \\
& \lambda_G = \lambda_{BU} = 40 + 0.045 \cdot 4000 = 220 \text{ \$/MWh} \\
& \lambda_{CE} = 20 + 0.045 \cdot 4000 = 200 \text{ \$/MWh}
\end{aligned}$$

Indeed the price in central Europe is lower than that of eastern Europe (Bulgaria and Greece), thus the assumption of the line being congested is consistent.

Question 4: We assume again that the Greek system produces. The following conditions must hold at equilibrium:

$$p_{CE} + p_G = 8000$$
$$20 + 0.045 p_{CE} = 40 + 0.045 p_G$$

From which we conclude that:

$$p_G = 8000 - p_{CE} \Rightarrow 20 + 0.045 p_{CE} = 40 + 0.045 \cdot (8000 - p_{CE}) \Rightarrow 0.09 p_{CE} = 380$$
$$\Rightarrow p_{CE} = 4222 \text{ MW}, p_G = 8000 - 4222 = 3778 \text{ MW}$$
$$\Rightarrow \lambda_G = \lambda_{CE} = 20 + 0.045 \cdot 4222 = 40 + 0.045 \cdot 3778 = 210 \text{ \$/MWh}$$

We further confirm that the line from central Europe to Bulgaria is not congested.

Question 5: In this case we would have

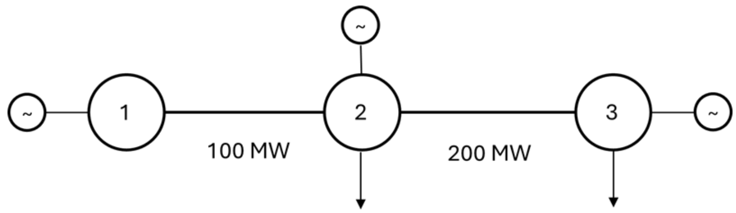
$$\lambda_G = 40 + 0.045 \cdot 8000 = 400 \text{ \$/MWh}$$

Question 6: The high prices are due to high demand in the summer (question 2), the failure of Kozloduy (question 3), and the congestion from central to eastern Europe (question 4). The existence of markets allows Greece to keep prices lower due to the import of cheaper energy from other countries (question 5).

Problem 5.3: The system of figure 5.3 consists of three nodes with generators in nodes 1, 2 and 3, and loads in nodes 2 and 3. The capacities of lines 1-2 and 2-3 are presented in the figure.
Compute the locational marginal prices, the production and consumption at each node, the net injection at each node, and the power flow on each transmission line at the optimal power flow solution, for the following input data (where Q is measured in MWh):

- Marginal cost of producers in node 1: $MC_1(Q) = 10 + 0.05Q$ \$/MWh
- Marginal cost of producers in node 2: $MC_2(Q) = 42 + 0.025Q$ \$/MWh
- Marginal cost of producers in node 3: $MC_3(Q) = 50 + 0.1Q$ \$/MWh

Figure 5.3: The network of problem 5.3.



- The consumers in node 2 have an inelastic demand of 150 MWh
- Marginal benefit of consumers in node 3: $MB_3(Q) = 80 - 0.1Q$ \$/MWh

Compute the following payments:

- From the system operator to the producers of node 1
- From the system operator to the producers of node 2
- From the system operator to the producers of node 3
- From the consumers of node 2 to the system operator
- From the consumers of node 3 to the system operator
- The congestion rent collected by the system operator

We recall that a financial transmission right (FTR) of one MW from node m to node n pays the difference of the locational marginal price between node m and node n. Suppose that the system operator issues 50 MW of FTRs from node 1 to node 2, and 120 MW of FTRs from node 2 to node 3.

- What is the payment of the transmission system operator for settling the FTRs?
- Is the congestion rent sufficient to cover the settlement of the FTRs?

Solution: Let us assume that both lines are congested. Then the following conditions need to hold for node 1:

$$10 + 0.05p_1 = \rho_1$$
$$p_1 = 100$$

The price at node 1 is thus $\rho_1 = 15$ \$/MWh.
The following conditions need to hold for node 2:

$$42 + 0.025p_2 = \rho_2$$
$$p_2 + 100 = 150 + 200$$

The production of generator 2 is $p_2 = 250$ MWh and the price at node 2 is thus $\rho_2 = 48.25$ \$/MWh.
The following conditions need to hold at node 3:

$$50 + 0.1p_3 = \rho_3$$
$$80 - 0.1d_3 = \rho_3$$
$$200 + p_3 = d_3$$

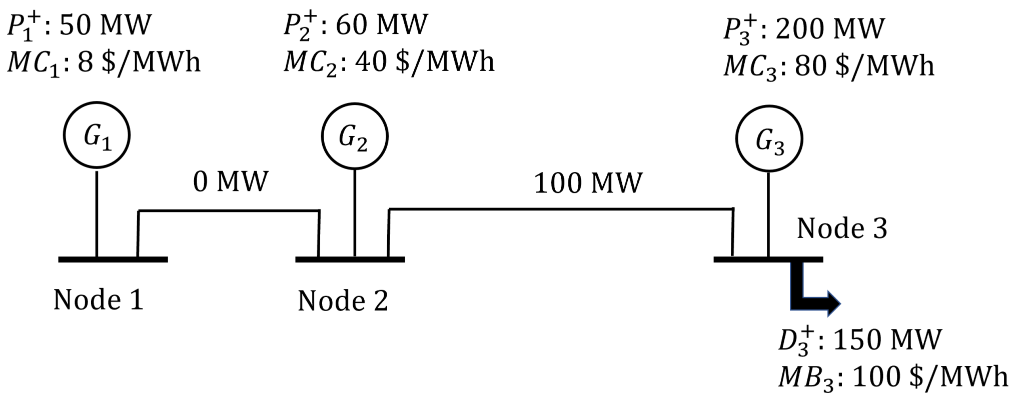
The third system is expressed equivalently as follows:

$$200 + p_3 = d_3$$
$$50 + 0.1p_3 = 80 - 0.1d_3 \Rightarrow 50 + 0.1p_3 = 80 - 0.1 \cdot (200 + p_3) = 60 - 0.1p_3$$
$$\Rightarrow p_3 = 50$$

From this we can conclude that $p_3 = 50$ MWh, $d_3 = 250$ MWh, $\rho_3 = 50 + 0.1 \cdot 50 = 80 - 0.1 \cdot 250 = 55$ \$/MWh. We observe that the primal and dual solutions are consistent with the profit maximization of all agents (including the system operator), thus the solution is valid.
Payments are as follows:

- From the system operator to the producers of node 1: $(100MWh) \cdot$ (15 \$/MWh) = \$1500
- From the system operator to the producers of node 2: $(250MWh) \cdot$ (48.25 \$/MWh) = \$12062.5
- From the system operator to the producers of node 3: $(50MWh) \cdot$ (55 \$/MWh) = \$2750
- From the consumers of node 2 to the system operator: $(150MWh) \cdot$ (48.25 \$/MWh) = \$7237.5
- From the consumers of node 3 to the system operator: $(250MWh) \cdot$ (55 \$/MWh) = \$13750

Figure 5.4: The system of problem 5.4.



- The congestion rent of the system operator: $13750 + 7237.5 - 1500 - 12062.5 - 2750 = \$4675$

The settlement of FTRs requires $(50\text{ MWh}) \cdot (\rho_2 - \rho_1) + (120\text{ MWh}) \cdot (\rho_3 - \rho_2) = (50\text{ MWh}) \cdot (48.25 - 15\text{ \$/MWh}) + (120\text{ MWh}) \cdot (55 - 48.25\text{ \$/MWh}) = \$2472.5$. Comparing to the congestion rent, we observe that the congestion rent suffices for covering the settlement of the FTRs, which we already knew would be the case because the FTRs that have been issued are consistent with the network limits.

Problem 5.4

Question 1: Compute the optimal solution (production p, consumption d, net injections r, flows f) for the system of figure 5.4 and the locational marginal prices which correspond to the optimal solution.

Question 2: Compute consumer benefit (CB) and production cost (PC) at the optimal solution. Compute producer surplus (PS), consumer surplus (CS) and congestion revenue (CR) at the optimal solution. How does welfare $W = CB - PC$ compare to the sum $PS + CS + CR$ (i.e. is W greater than, less than or equal to $PS + CS + CR$).

Solution

Question 1: The optimal solution is as follows:

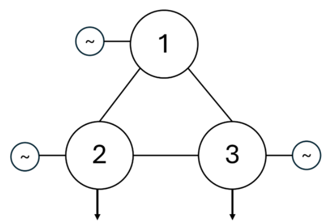
- Production $p_1 = 0$ MW, $p_2 = 60$ MW, $p_3 = 90$ MW
- Consumption $d_3 = 150$ MW
- Net injections $r_1 = 0$ MW, $r_2 = 60$ MW, $r_3 = -60$ MW
- Flows $f_{1-2} = 0$ MW, $f_{2-3} = 60$ MW

Locational marginal prices are not unique. We pick $\rho_1 = 8$ \$/MWh, $\rho_2 = 80$ \$/MWh, $\rho_3 = 0$ \$/MWh.
Question 2: Consumer benefit is $CB = 100 \cdot 150 = 15000$ \$. Production cost is $PC = 40 \cdot 60 + 80 \cdot 90 = 2400 + 7200 = 9600$ \$. Welfare is $W = CB - PC = 15000 - 9600 = 5400$ \$.
Producer surplus is $PS = (80 - 40) \cdot 60 = 2400$ \$.
Consumer surplus is $CS = (100 - 80) \cdot 150 = 3000$ \$.
Congestion rent is $CR = 0$ \$.
We have that $PS + CS + CR = 2400 + 3000 = 5400$ \$. We observe that this is equal to W. This is not a coincidence, but holds by definition: if we develop $PS + CS + CR$ the terms which implicate price cancel each other out and only consume benefit minus production cost remains. Prices are just a way to split the overall "pie".

Problem 5.5: The three-node network of figure 5.5 has the following characteristics:

- Lines have identical electrical characteristics (susceptances)
- The generator of node 1 has a marginal cost of 80 \$/MWh and unlimited capacity
- The generator of node 2 has a marginal cost of 160 \$/MWh and unlimited capacity
- The generator of node 3 has a marginal cost of 280 \$/MWh and unlimited capacity
- There is an inelastic load of 200 MW in node 2

Figure 5.5: The network of problem 5.5.



- There is an inelastic load of 400 MW in node 3
- Line 1-2 has a thermal limit of 100 MW
- Line 2-3 has a thermal limit of 200 MW
- Line 1-3 has a thermal limit of 200 MW

Question 1: Compute the PTDF coefficients of the system, assuming that node 3 is the reference node.

Question 2: Compute the optimal dispatch (production, network injections and flows which minimize cost), and the production cost at the optimal solution. You can use the graphical solution on the two dimensions p_1 and p_2 to solve the problem, and only account for the constrains on line flows in the direction 1-to-3, 1-to-2, 2-to-3, as well as the constraint that the production of unit 3 is non-negative.

Question 3: Compute locational marginal prices (LMPs). Attention: for the price of node 3, use sensitivity analysis to show that the locational marginal price is not unique and compute the range of locational marginal prices.

Solution

Question 1: The PTDF matrix is:

$$PTDF = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

The lines of the table correspond to lines 1-2, 1-3, 2-3. The columns correspond to nodes 1, 2.

Question 2: Power balance requires that

$$p_1 + p_2 + p_3 = 600$$

Thus, expressing the objective function as a function of the variables p_1 and p_2 we have

$$\begin{aligned}
&80 \cdot p_1 + 160 \cdot p_2 + 280 \cdot p_3 = \\
&80 \cdot p_1 + 160 \cdot p_2 + 280 \cdot (600 - p_1 - p_2)s = \\
&168000 - 200 \cdot p_1 - 120 \cdot p_2
\end{aligned}$$

The feasible set is described by the following inequalities:

$$\begin{aligned}
\text{Line 1-2 :}\quad & -100 \leq \frac{1}{3}p_1 - \frac{1}{3}(p_2 - 200) \leq 100 \Leftrightarrow -500 \leq p_1 - p_2 \leq 100 \\
\text{Line 1-3 :}\quad & -200 \leq \frac{2}{3}p_1 + \frac{1}{3}(p_2 - 200) \leq 200 \Leftrightarrow -400 \leq 2 \cdot p_1 + p_2 \leq 800 \\
\text{Line 2-3 :}\quad & -200 \leq \frac{1}{3}p_1 + \frac{2}{3}(p_2 - 200) \leq 200 \Leftrightarrow -200 \leq p_1 + 2 \cdot p_2 \leq 1000 \\
& p_3 \geq 0 \Rightarrow -p_1 - p_2 \geq 0
\end{aligned}$$

The feasible set is presented in figure 5.6. The blue points are extreme points of the feasible set. The point at which lines 1-2 and 1-3 are simultaneously congested satisfies the following two equations:

$$\begin{aligned}
&p_1 - p_2 = 100 \Rightarrow p_1 = 100 + p_2 \\
&2 \cdot p_1 + p_2 = 800 \Rightarrow 2 \cdot (100 + p_2) + p_2 = 800 \Rightarrow p_2 = 200 \Rightarrow p_1 = 300
\end{aligned}$$

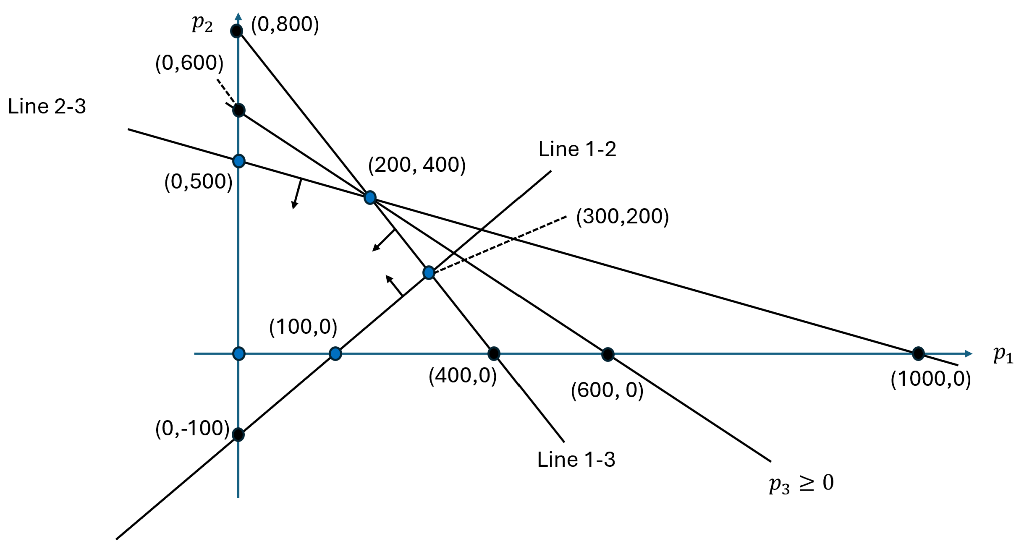
This point leads to a flow on line 2-3 which is equal to

$$\frac{1}{3}300 + \frac{2}{3}(200 - 200) = 100$$

Thus this flow is acceptable for line 2-3.
Lines 1-3 and 2-3 are simultaneously congested at the point of operation which solves the following equations:

$$\begin{aligned}
&2 \cdot p_1 + p_2 = 800 \Rightarrow p_2 = 800 - 2 \cdot p_1 \\
&p_1 + 2 \cdot p_2 = 1000 \Rightarrow p_1 + 2 \cdot (800 - 2 \cdot p_1) = 1000 \Rightarrow p_1 = \frac{600}{3} = 200 \Rightarrow p_2 = 400
\end{aligned}$$

Figure 5.6: The feasible set of problem 5.5.



We substitute the four extreme points in the objective function in order to find the one with the lowest cost:

Point (0,0) attains a cost of 168000.

Point (0, 500) attains a cost of $168000 - 200 \cdot p_1 - 120 \cdot p_2 = 168000 - 200 \cdot 0 - 120 \cdot 500 = 108000$.

Point (200, 400) attains a cost of $168000 - 200 \cdot p_1 - 120 \cdot p_2 = 168000 - 200 \cdot 200 - 120 \cdot 400 = 80000$.

Point (300, 200) attains a cost of $168000 - 200 \cdot p_1 - 120 \cdot p_2 = 168000 - 200 \cdot 300 - 120 \cdot 200 = 84000$.

Point (100, 0) attains a cost of $168000 - 200 \cdot p_1 - 120 \cdot p_2 = 168000 - 200 \cdot 100 - 120 \cdot 0 = 148000$.

The optimal solution is (200, 400). Thus the optimal dispatch is $(p_1, p_2, p_3) = (200, 400, 0)$. The flow is $(f_{1-2}, f_{2-3}, f_{1-3}) = (\frac{1}{3}p_1 - \frac{1}{3}(p_2 - 200), 200, 200) = (0, 200, 200)$.

Cost has already been computed and is equal to 80000 \$.

Question 3: The locational marginal prices of nodes 1 and 2 are equal to 80 \$/MWh and 160 \$/MWh, since there are units in hese locations which produce a positive quantity below their technical maximum.

The locational marginal price of location 3 is equal to the increase in cost when we increase demand in node 3 by 1 MW. The constraint $p_3 \geq 0$ is no longer binding, see figure 5.7. Since lines 1-3 and 2-3 remain at their limit, the optimal production of units 1 and 2 does not change, and the additional demand is covered by unit 3, thus the additional cost is equal to the marginal cost of the unit in node 3, i.e. 280 \$/MWh.

The locational marginal price of node 3 is also equal to the cost <u>*reduction*</u> if demand in node 3 is <u>*decreased*</u> by 1 MW. Concretely, what changes in the analysis is that now the extreme point (200, 400) "breaks into" two extreme points, as shown in figure 5.8. The firs of the two new extreme points satisfies the following equations:

$$p_1 + p_2 = 599$$
$$p_1 + 2 \cdot p_2 = 1000$$

The solution of this system is $(p_1, p_2) = (198, 401)$. The cost is $80 \cdot p_1 + 160 \cdot p_2 + 280 \cdot p_3 = 80 \cdot 198 + 160 \cdot 401 + 280 \cdot 0 = 80000$.

Figure 5.7: Change in the feasible set of problem 5.5 when demand in node 3 is increased by 1 MW.

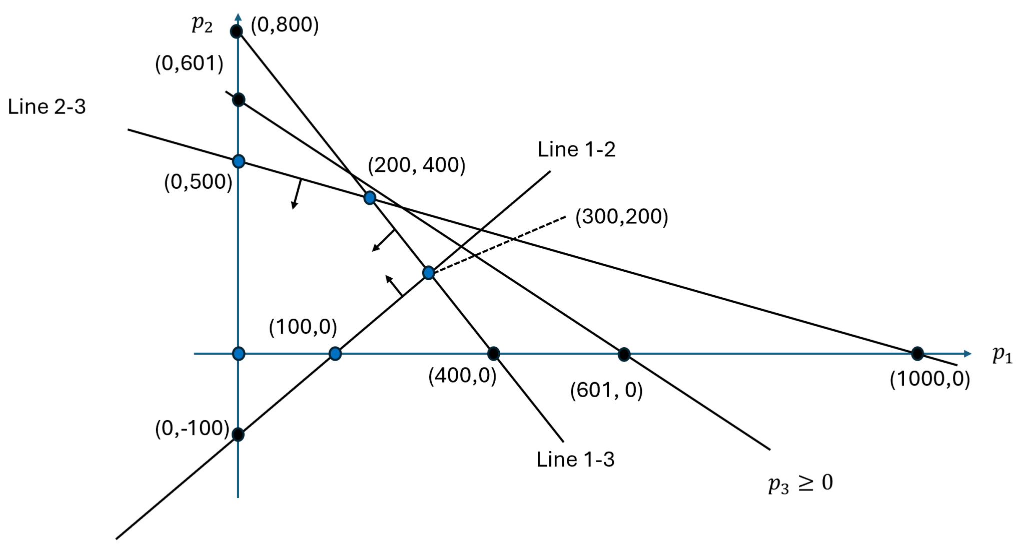


Figure 5.8: Change in the feasible set of problem 5.5 when demand in node 3 is decreased by 1 MW.

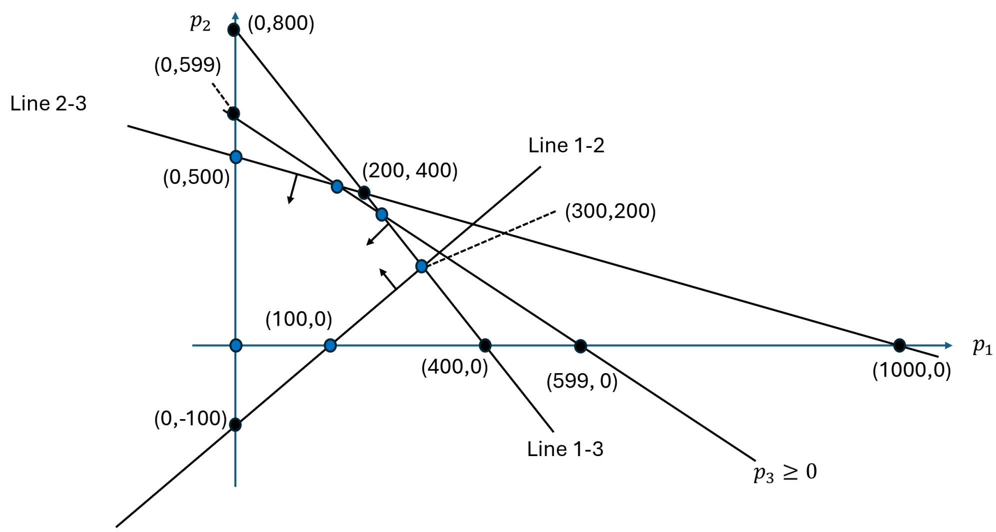
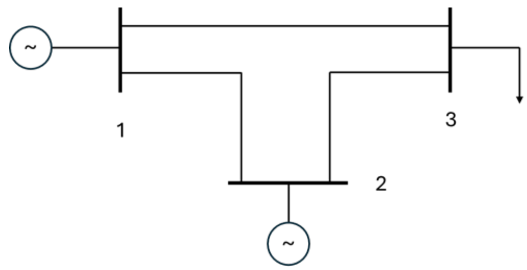


Figure 5.9: The system of problem 5.6.



The second new extreme point satisfies the following equations:

$$p_1 + p_2 = 599$$
$$2 \cdot p_1 + p_2 = 800$$

The solution of this system of equations is $(p_1, p_2) = (201, 398)$. The cost is $80 \cdot p_1 + 160 \cdot p_2 + 280 \cdot p_3 = 80 \cdot 201 + 160 \cdot 398 + 280 \cdot 0 = 79760$. Thus, the second point is optimal, and leads to a cost reduction of 240 \$/MWh, which is the locational marginal price of node 3.

Problem 5.6: The system of figure 5.9 consists of three nodes with generators in nodes 1 and 2 and loads in node 3.

Question 1: Compute the power transfer distribution factors (PTDFs) given that node 3 is the hub node, and given the following per unit values for the susceptance of each line:

$$X_{1-2} = 1, X_{2-3} = 1, X_{1-3} = 0.5$$

Question 2.1: Consider a system of contract paths where line usage rights are allocated based on the physical capacity of lines, and let us assume that line capacities are the following:

$$T_{1-2} = 500 \text{ MW}, T_{1-3} = 500 \text{ MW}, T_{2-3} = 500 \text{ MW}$$

Let us assume that the generator of node 1 wishes to engage in a forward contract with the load of node 3, and procures 500 MW along path 1-3, which means that the counter-parties procure a right to inject 500 MW in node 1 (for the producer of node 1) and a right to withdraw 500 MW in node 3 (for the load in node 3). Is this contract path compatible with the limits of the network?

Question 2.2: Let us assume that, in addition to the contract paths of question 2.1, the generator of node 2 wishes to engage in a forward contract path with the load of node 3, and procures 500 MW along path 2-3, which means that the counter-parties gain a right to inject 500 MW in node 2 (for the producer in node 2) and to withdraw 500 MW in node 3 (for the load in node 3). Is this combination of contract paths of questions 2.1 and 2.2 compatible with the limits of the network?

Question 3: Let us assume that the load in node 3 is 800 MW, and that the generators submit the following offers:

- Generator 1: 1000 MW at 40 $/MWh

- Generator 2: 1000 MW at 80 $/MWh

Question 3.1: Compute the optimal power flow (production p of units 1 and 2, injections r at all nodes and flows f on all lines) and the locational marginal prices (LMPs) ρ. The contract paths of question 2 should be ignored for this question. Hint: you can use the graphical method for solving linear programs, accounting only for the constraint of line 1-3 in the direction from node 1 o node 3 (in the sense that this constraint should be accounted for, but without this meaning that it is necessarily binding at the optimal solution).

Question 3.2: What is the production cost? What is the producer profit? What is the congestion rent?

Solution

Question 1: The PTDFs are computed as follows:

$$\begin{aligned}
PTDF_{1-3,1} &= \frac{X_{1-2}+X_{2-3}}{X_{1-2}+X_{2-3}+X_{1-3}} = \frac{2}{2.5} = 0.8 \\
PTDF_{1-2,1} &= \frac{X_{1-3}}{X_{1-2}+X_{2-3}+X_{1-3}} = \frac{0.5}{2.5} = 0.2 \\
PTDF_{2-3,1} &= \frac{X_{1-3}}{X_{1-2}+X_{2-3}+X_{1-3}} = \frac{0.5}{2.5} = 0.2 \\
PTDF_{1-3,2} &= \frac{X_{2-3}}{X_{1-2}+X_{2-3}+X_{1-3}} = \frac{1}{2.5} = 0.4 \\
PTDF_{1-2,2} &= -\frac{X_{2-3}}{X_{1-2}+X_{2-3}+X_{1-3}} = -\frac{1}{2.5} = -0.4 \\
PTDF_{2-3,2} &= \frac{X_{1-2}+X_{1-3}}{X_{1-2}+X_{2-3}+X_{1-3}} = \frac{1.5}{2.5} = 0.6
\end{aligned}$$

Question 2.1: The flows that result from the contract paths are:

$$\begin{aligned}
f_{1-2} &= PTDF_{1-2,1} \cdot r_1 + PTDF_{1-2,2} \cdot r_2 = 0.2 \cdot 500 + (-0.4) \cdot 0 = 100 \text{ MW} \\
f_{1-3} &= PTDF_{1-3,1} \cdot r_1 + PTDF_{1-3,2} \cdot r_2 = 0.8 \cdot 500 + 0.4 \cdot 0 = 400 \text{ MW} \\
f_{2-3} &= PTDF_{2-3,1} \cdot r_1 + PTDF_{2-3,2} \cdot r_2 = 0.2 \cdot 500 + 0.6 \cdot 0 = 100 \text{ MW}
\end{aligned}$$

The flows are indeed compatible with the limits of the network.

Question 2.2: The flows that result from the contract paths are:

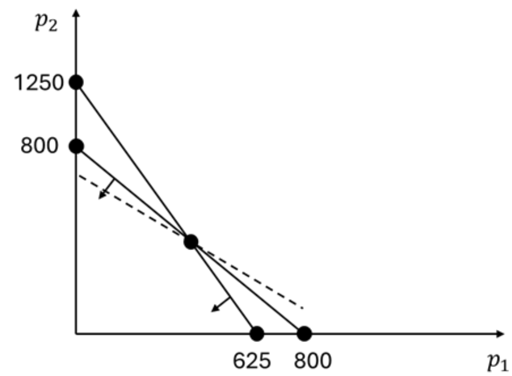
$$\begin{aligned}
f_{1-2} &= PTDF_{1-2,1} \cdot r_1 + PTDF_{1-2,2} \cdot r_2 = 0.2 \cdot 500 + (-0.4) \cdot 500 = -100 \text{ MW} \\
f_{1-3} &= PTDF_{1-3,1} \cdot r_1 + PTDF_{1-3,2} \cdot r_2 = 0.8 \cdot 500 + 0.4 \cdot 500 = 600 \text{ MW} \\
f_{2-3} &= PTDF_{2-3,1} \cdot r_1 + PTDF_{2-3,2} \cdot r_2 = 0.2 \cdot 500 + 0.6 \cdot 500 = 400 \text{ MW}
\end{aligned}$$

The flows are not compatible with the limits of the network, concretely the limit of line 1-3 is violated.

Question 3.1: We solve the following optimization problem:

$$\begin{aligned}
&\min_{p_1,p_2} 40 \cdot p_1 + 100 \cdot p_2 \\
&800 - p_1 - p_2 = 0 \\
&0.8 \cdot p_1 + 0.4 \cdot p_2 \leq 500 \\
&p \geq 0
\end{aligned}$$

Figure 5.10: The feasible set of question 3.1 of problem 5.6.



The constraints are represented graphically in figure 5.10. The dashed iso-cost corresponds to the objective function value at the optimal solution.
The optimal solution corresponds to the intersection of the following linear equations:

$$p_1 + p_2 = 800$$
$$0.8 \cdot p_1 + 0.4 \cdot p_2 = 500$$

Substituting the first equation into the second:

$$p_2 = 800 - p_1$$
$$0.8 \cdot p_1 + 0.4 \cdot (800 - p_1) = 500 \Rightarrow$$
$$0.4 \cdot p_1 = 500 - 0.4 \cdot 800 \Rightarrow$$
$$p_1 = 450 \Rightarrow p_2 = 350$$

Thus

$$r_1 = 450, r_2 = 350, r_3 = -800$$

And

$$\begin{aligned} f_{1-2} &= PTDF_{1-2,1} \cdot r_1 + PTDF_{1-2,2} \cdot r_2 = 0.2 \cdot 450 + (-0.4) \cdot 350 = -50 \text{ MW} \\ f_{1-3} &= PTDF_{1-3,1} \cdot r_1 + PTDF_{1-3,2} \cdot r_2 = 0.8 \cdot 450 + 0.4 \cdot 350 = 500 \text{ MW} \\ f_{2-3} &= PTDF_{2-3,1} \cdot r_1 + PTDF_{2-3,2} \cdot r_2 = 0.2 \cdot 450 + 0.6 \cdot 350 = 300 \text{ MW} \end{aligned}$$

The LMPs of nodes 1 and 2 are the marginal costs of the marginal units, thus

$$\rho_1 = 40, \rho_2 = 80$$

For node 3 we use sensitivity analysis, concretely the power balance constraint is shifted upwards by 1 MW. Thus the optimal solution becomes the intersection of

$$\begin{aligned} &p_1 + p_2 = 801 \\ &0.8 \cdot p_1 + 0.4 \cdot p_2 = 500 \end{aligned}$$

Substituting the first equation into the second:

$$\begin{aligned} &p_2 = 801 - p_1 \\ &0.8 \cdot p_1 + 0.4 \cdot (801 - p_1) = 500 \Rightarrow \\ &0.4 \cdot p_1 = 500 - 0.4 \cdot 801 \Rightarrow \\ &p_1 = 449 \Rightarrow p_2 = 352 \end{aligned}$$

The new cost is

$$40 \cdot p_1 + 80 \cdot p_2 = 40 \cdot 449 + 80 \cdot 352 = 46120$$

The original cost is

$$40 \cdot p_1 + 80 \cdot p_2 = 40 \cdot 450 + 80 \cdot 350 = 46000$$

Thus the price of node 3 is $\rho_3 = 120$.

Question 3.2: The production cost is computed in the previous question and is equal to 46000 \$.
Producer profit is 0 \$.
Congestion rent is

$$\rho_3 \cdot 800 - \rho_1 \cdot 450 - \rho_2 \cdot 350 = 120 \cdot 800 - 40 \cdot 450 - 80 \cdot 350 = 50000\$$$

Chapter 6

Ancillary services

Problem 6.1: Consider a problem of co-optimizing energy and downward reserve:

$$
\begin{aligned}
& \min_{p,r} \sum_{g \in G} MC_g p_g \\
(\gamma_g): \quad & -p_g + r_g \leq -P_g^- \\
(\lambda R): \quad & DR - \sum_{g \in G} r_g = 0 \\
(\lambda): \quad & D - \sum_{g \in G} p_g = 0 \\
& p, r \geq 0
\end{aligned}
$$

In this model, the decision variables are p_g for the production of energy from unit g, and r_g for the provision of downward reserve from unit g. The parameters are MC_g for the marginal cost of unit g, P_g^- for the technical minimum of unit g, DR for the downward reserve requirement, and D for the inelastic energy demand. The first constraint implies that a unit that produces p_g MWh of energy cannot offer more downward reserve than $p_g - P_g^-$ MWh, otherwise it risks violating its technical minimum if activated downward.

Question 1: Express the KKT conditions of the problem.

Question 2: Suppose that, in the optimal solution of the problem, there is a unit $\bar{g}$ with marginal cost $MC_{\bar{g}} = 20$ \$/MWh that offers energy $p_{\bar{g}} = 10$

MWh and downward reserve $r_{\bar{g}} = 10$ MWh, and that the price of downward reserve λR is equal to 10 \$/MWh. Compute the energy price λ.

Question 1: The KKT conditions of the problem are expressed as follows:

$$DR - \sum_{g \in G} r_g = 0$$
$$D - \sum_{g \in G} p_g = 0$$
$$0 \leq \gamma_g \perp p_g - r_g - P_g^- \geq 0, g \in G$$
$$0 \leq p_g \perp MC_g - \gamma_g - \lambda \geq 0, g \in G$$
$$0 \leq r_g \perp \gamma_g - \lambda R \geq 0, g \in G$$

Question 2: We know that $p_{\bar{g}} > 0$, thus $\gamma_{\bar{g}} = MC_{\bar{g}} - \lambda$. Moreover, $r_{\bar{g}} > 0$, thus $\gamma_{\bar{g}} = \lambda R$. Equating:

$$\lambda R = MC_{\bar{g}} - \lambda \Rightarrow \lambda = MC_{\bar{g}} - \lambda R = 20 - 10 = 10 \text{ \$/MWh}.$$

Problem 6.2: Consider a day-ahead market which co-optimizes energy and reserves, with the following characteristics:

- Load: 15 MW
- Upward reserve requirement: 15 MW
- Generator 1 with a capacity of 25 MW and a marginal cost of 10 \$/MWh, which cannot offer reserve (inflexible)
- Generator 2 with a capacity of 20 MW and a marginal cost of 500 \$/MWh, can offer reserve (flexible)
- Generator 3 with a capacity of 10 MW and a marginal cost of 100 \$/MWh, can offer reserve (flexible)

Question 1: Compute the outcome of the day-ahead market, specifically the energy and reserve dispatch of each unit, as well as the prices of energy and reserves.

Question 2: Let us assume that in real time there is a negative imbalance of 8 MW, and let us also assume that in the balancing market free bids are allowed, which means that even units that have not been cleared for reserves can participate in the balancing market. What is the balancing price, and what is the dispatch of each unit in real time? What is the system cost in real time?

Question 3: Identical to question 2, but without allowing for free bids in the balancing market.

Solution
Question 1: The idea is that more expensive flexible units are used for covering reserves, and then cheaper (flexible and non-flexible) units are used for serving energy. Concretely:

- Generator 1: 15 MW energy
- Generator 2: 15 MW reserves, 0 MW energy
- Unit 3: 0 MW reserves, 0 MW energy

The energy price is equal to the marginal cost of unit 1, i.e. 10 \$/MWh. The price of reserve is 0 \$/MWh.

Question 2: The dispatch of each unit is as follows:

- Unit 1: 15 MW
- Unit 2: 0 MW
- Unit 3: 8 MW

The balancing price is equal to the marginal cost of the marginal unit, i.e. unit 3, thus the price is 100 \$/MWh. The operating cost of the system is $(15 \text{ MWh}) \cdot (10 \text{ \$/MWh}) + (8 \text{ MWh}) \cdot (100 \text{ \$/MWh}) = 950$ \$.

Question 3: The dispatch of each unit is as follows:

- Unit 1: 15 MW
- Unit 2: 8 MW

Table 6.1: The table of exercise 6.3.

Technology	Marginal cost ($/MWh)	Installed capacity (MW)
Nuclear	6.5	200
Coal	25	400
Natural gas	80	300

- Unit 3: 0 MW

The balancing price is equal to the marginal cost of the marginal unit, i.e. unit 2, thus the price is 500 $/MWh. The operating cost of the system is (15 MWh) · (10 $/MWh) + (8 MWh) · (500 $/MWh) = 4150 $.

Problem 6.3: Consider a system with three technologies, which are shown in table 6.1
Suppose that imbalances in the system are distributed uniformly in the interval [0, 400] MW. We assume that the VOLL of consumers is $VOLL = 1000$ $/MWh.

Question 1: What is the equilibrium price for an energy-only market when load is 800 MW?

Question 2: Compute analytically (as a function of x) the ORDC when it is given by the following formula

$$MR(x) = VOLL \cdot LOLP(x).$$

Question 3: What is the optimal dispatch of energy and reserves (d, dr, p, r) and the equilibrium energy and reserve price (λ and μ) in a market with co-optimization of energy and reserves (($ORDC$) model) when load is 800 MW?

Solution

Question 1: We solve an economic dispatch, so the price is set by natural gas at 80 $/MWh.

Question 2: We have that

$$LOLP(x) = \mathbb{P}[Imb > x] = \begin{cases} 0, & x > 400 \\ \frac{400-x}{400}, & x \leq 400 \end{cases}$$

Thus

$$MR(x) = \begin{cases} 0, & x > 400 \\ 1000 - 2.5 \cdot x, & x \leq 400 \end{cases}$$

Question 3: The idea is that with an installed capacity of 900 MW the reserve will only be partially covered, by 100 MW, thus the price of reserve will be equal to the valuation of MR at 100 MW, thus $\mu = MR(100) = 1000 - 2.5 \cdot 100 = 750$ \$/MWh. The energy price is given by the no-arbitrage conditions of the marginal producer, which is the natural gas producer, thus $\lambda = \mu + MC_{gas} = 750 + 80 = 830$ \$/MWh.
The energy demand is $d = 800$ MW, the reserve demand is $dr = 100$ MW, the energy production decisions are $p_{nuc} = 200$, $p_{coal} = 400$, $p_{gas} = 200$ MW, the reserve allocations are $r_{nuc} = 0$, $r_{coal} = 0$, $r_{gas} = 100$ MW.

Chapter 7

Unit commitment

Problem 7.1: Consider a system with the production units of table 7.1. The load of the system for the next 24 hours is described in table 7.2.

1. Implement a unit commitment model which minimizes the cost of operating the system. Assume that all units are off at the end of the previous day. What is the minimum cost at which the system operates?

2. Introduce a reserve requirement of 200 MW for an upward reserve product with a full activation time of one hour. What is the new minimum cost of operation? How do you explain the new cost, relative to the one of the previous question?

Solution: The code is available in the textbook website.

1. The minimum cost of operation is $1587980.

2. The minimum cost of operation is $1596725. The cost has increased, because the model with zero reserve requirements (which is the model of the previous section) is a relaxation of the model with non-zero reserve requirements.

An alternative formulation of the second model includes reserve provision in the ramp up constraints [1].

Problem 7.2: Prove that there is no pure strategy Nash equilibrium in example 7.5.

Table 7.1: The generators of problem 7.1.

Unit	PMax (MW)	PMin (MW)	Ramp (MW/min)	UT (hrs)
G1	200	50	2	3
G2	300	75	5	2
G3	250	50	1	6
G4	800	100	8	1
G5	600	80	5	5
Unit	**DT (hrs)**	**MC ($/MWh)**	**SUC ($)**	**MLC ($)**
G1	2	28	10000	1000
G2	4	52	5000	3000
G3	3	27	7000	4000
G4	6	39	20000	2500
G5	5	85	3000	3000

Table 7.2: The load of problem 7.1.

Hour	Load (MW)	Hour	Load (MW)	Hour	Load (MW)
1	600	9	1850	17	1500
2	390	10	1750	18	1750
3	410	11	1600	19	1800
4	520	12	1750	20	1900
5	615	13	1700	21	1800
6	750	14	1750	22	1600
7	1200	15	1600	23	1050
8	1500	16	1400	24	780

Solution: We consider a game in which agents choose at what price to bid their capacity P^+ in the auction. Denote the fixed cost of the agents as K, their marginal cost as MC and their max capacity as P^+. Denote δ as the remainder of demand that is covered by the unique unit which is dispatched partially. Denote p as the presumed price at which everyone is bidding.
In order to validate that there is a Nash equilibrium, we consider the three strategies of the agents:

- Bid above p. Then the payoff is zero, because the unit is not cleared.
- Bid below p. Then the payoff is $(p - MC) \cdot P^+ - K$.
- Bid at p. Then the expected payoff, since there are three generators in the system, is

$$\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot ((p - MC) \cdot P^+ - K) + \frac{1}{3} \cdot ((p - MC) \cdot (P^+ - \delta) - K)$$

In order for a pure strategy to be an equilibrium, it must be that the price be high enough to cover fixed costs:

$$\begin{aligned} &(p - MC)P^+ - K \geq 0 \Leftrightarrow \\ &p \geq MC + \frac{K}{P^+} \end{aligned}$$

And it must be that the third action be better than the second one:

$$\begin{aligned} &\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot ((p - MC) \cdot P^+ - K) + \frac{1}{3} \cdot ((p - MC) \cdot (P^+ - \delta) - K) \geq \\ &(p - MC)P^+ - K \Leftrightarrow \\ &\frac{1}{3} \cdot ((p - MC) \cdot P^+ - K) \leq -\frac{1}{3}(p - MC) \cdot \delta \Leftrightarrow \\ &p \leq MC + \frac{K}{\delta + P^+} \end{aligned}$$

Since $\frac{K}{\delta + P^+} < \frac{K}{P^+}$, the two inequalities cannot hold simultaneously, thus there cannot be a pure strategy Nash equilibrium.
The intuition of the result can be understood by reasoning by contradiction. If there were a pure strategy Nash equilibrium, then the market price should be such that a unit can cover its marginal cost plus its fixed cost amortized over its entire production. But if that is the case, then it is advantageous

for a unit to unilaterally underbid its competitors, because in that way it is guaranteed to be always fully dispatched, whereas by bidding at the price of all competitors it is sometimes partially dispatched. Note that Stoft discusses the nonexistence in page 246 of his book [3].

Problem 7.3: Consider a market with the following orders:

- Order A: consume 90 MWh at 100 $/MWh
- Order B: consume 100 MWh at 20 $/MWh
- Order C: produce 100 MWh at 30 $/MWh
- Order D: produce 90 MWh at 90 $/MWh

1. What is the optimal solution to the economic dispatch problem, and the market clearing price? What is the producer profit, the consumer surplus, and the social surplus?
2. Suppose that order C is a block order (i.e. either it is accepted for its entire quantity or rejected). What is the maximum surplus that we can generate from clearing this market? How does it compare to the surplus of the first question?
3. Is there an equilibrium price in the market of question 2? If there is, specify it. If not, justify your answer, explaining whether we face an excess or shortage in supply for different candidate market clearing prices.
4. Compute the lost opportunity cost (LOC) of every order for a market price of 30 $/MWh, and the total lost opportunity cost of the market.

Solution

Question 1: The market clears at 30 $/MWh. The optimal solution is the following:

- Order A consumes 90 MWh
- Order B consumes 0 MWh

- Order C produces 90 MWh
- Order D produces 0 MWh

Producer profit is equal to 90 MWh · (0 $/MWh) = $0. The consumer surplus is 90 MWh · (100 - 30 $/MWh) = $6300. The social surplus is $6300.

Question 2: The optimal solution is the following:

- Order A consumes 90 MWh
- Order B consumes 10 MWh
- Order C produces 90 MWh
- Order D produces 10 MWh

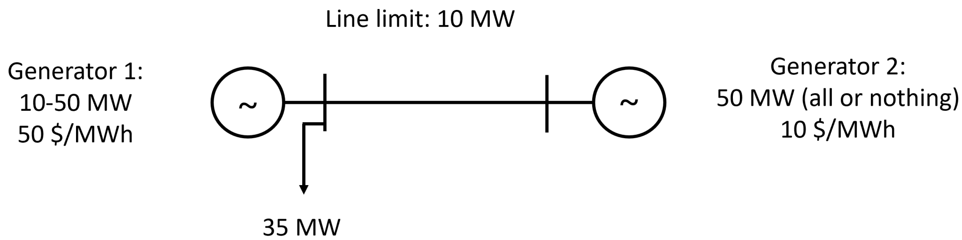
The surplus is (90 MWh · 100 $/MWh) + (10 MWh · 20 $/MWh) − (100 MWh · 30 $/MWh) = $6200. Thus the surplus is $100 less than that of question 1.

Question 3: No. For a price below 30 $/MWh we face under-supply (demand > supply). For a price above 30 $/MWh we face over-supply (supply > demand). For a price of 30 $/MWh we face over-supply if we reject the (indifferent) order C and over-supply if we accept it.

Question 4:

- Order A decides the same at the market optimal solution and at the solution of question 2 (consume 90 MWh), thus its LOC is equal to 0.
- Order B consumes 10 MWh at the market optimal solution, resulting in a surplus of 10 MWh · (-10 $/MWh) = -$100, whereas the agent would prefer to consume 0 MWh resulting in a surplus of $0, thus the LOC is equal to $100.
- Order C produces the same at the market optimal solution at the solution of question 2 (produce 100 MWh), thus the LOC is equal to 0.

Figure 7.1: The system of problem 7.4.



- Order D producers the same at the market optimal solution and the solution of question 2 (produce 0 MWh), thus the LOC is equal to 0.

The total LOC of the market is equal to \$100.

Problem 7.4: Consider the two-generator system which is presented in figure 7.1.
The market clearing problem is formulated as follows:

$$\begin{aligned}
&\min_{p,u,f} 50 \cdot p_1 + 10 \cdot p_2 \\
&35 + f - p_1 = 0 \\
&-f - p_2 = 0 \\
&f \leq 10 \\
&-10 \leq f \\
&p_1 \leq 50 \cdot u_1 \\
&10 \cdot u_1 \leq p_1 \\
&p_2 \leq 50 \cdot u_2 \\
&50 \cdot u_2 \leq p_2 \\
&p \geq 0, u \in \{0, 1\}
\end{aligned}$$

Question 1: Compute the optimal solution of the problem (i.e. the optimal values of all primal variables).

Question 2: The convex hull price is computed from the linear relaxation of the above problem.

$$
\begin{aligned}
& \min_{p,u,f} 50 \cdot p_1 + 10 \cdot p_2 \\
(\rho_1): \quad & 35 + f - p_1 = 0 \\
(\rho_2): \quad & -f - p_2 = 0 \\
(\lambda^+): \quad & f \leq 10 \\
(\lambda^-): \quad & -10 \leq f \\
(\mu_1^+): \quad & p_1 \leq 50 \cdot u_1 \\
(\mu_1^-): \quad & 10 \cdot u_1 \leq p_1 \\
(\mu_2^+): \quad & p_2 \leq 50 \cdot u_2 \\
(\mu_2^-): \quad & 50 \cdot u_2 \leq p_2 \\
(\delta_1): \quad & u_1 \leq 1 \\
(\delta_2): \quad & u_2 \leq 1 \\
& p \geq 0, u \geq 0
\end{aligned}
$$

Express the KKT conditions of the linear relaxation.

Question 3: Compute the primal optimal solution of the linear relaxation. Hint: you do not need the KKT conditions.

Question 4: Compute the dual optimal solution of the linear relaxation. Hint: you can use the KKT conditions and the solution of question 3.

Question 5: Compute the LOC of the two producers.

Question 6: Congestion revenue shortfall is defined in section 7.3.2 of the textbook as the LOC of the transmission system operator (where surplus maximization of the transmission system operator has been analyzed in problem 4.2 of this solutions manual). Compute the congestion revenue shortfall for the network of this problem as well as the total LOC of the system (producers and network operator).

Solution

Question 1: Generator 2, although cheaper, cannot operate, because demand exceeds the 50 MW which this unit can offer. Thus generator 1 will operate and there will be no flow on the network. Concretely:

$$p_1 = 35, u_1 = 1$$
$$p_2 = 0, u_2 = 0$$
$$f = 0$$

Question 2: The KKT conditions of the linear relaxation are expressed as follows:

$$\begin{aligned}
& 35 + f - p_1 = 0 \\
& -f - p_2 = 0 \\
& 0 \leq \lambda^+ \perp 10 - f \geq 0 \\
& 0 \leq \lambda^- \perp 10 + f \geq 0 \\
& 0 \leq \mu_1^+ \perp 50 \cdot u_1 - p_1 \geq 0 \\
& 0 \leq \mu_1^- \perp p_1 - 10 \cdot u_1 \geq 0 \\
& 0 \leq \mu_2^+ \perp 50 \cdot u_2 - p_2 \geq 0 \\
& 0 \leq \mu_2^- \perp p_2 - 50 \cdot u_2 \geq 0 \\
& 0 \leq \delta_1 \perp 1 - u_1 \geq 0 \\
& 0 \leq \delta_2 \perp 1 - u_2 \geq 0 \\
& 0 \leq p_1 \perp \mu_1^+ - \mu_1^- - \rho_1 + 50 \geq 0 \\
& 0 \leq p_2 \perp \mu_2^+ - \mu_2^- - \rho_2 + 10 \geq 0 \\
& 0 \leq u_1 \perp -50 \cdot \mu_1^+ + 10 \cdot \mu_1^- + \delta_1 \geq 0 \\
& 0 \leq u_2 \perp -50 \cdot \mu_2^+ + 50 \cdot \mu_2^- + \delta_2 \geq 0 \\
(f): \quad & \rho_1 - \rho_2 + \lambda^+ - \lambda^- = 0
\end{aligned}$$

Question 3: We wish to draw as much power as possible from the cheap unit, thus we choose $p_2 = 10$, and the rest of the power is drawn from unit 1. The flow is 10 MW from node 2 to node 1. An optimal (not necessarily unique) solution is therefore the following:

$$p_1 = 25, u_1 = \frac{25}{50} = 0.5$$
$$p_2 = 10, u_2 = \frac{10}{50} = 0.2$$
$$f = -10$$

Question 4: From the primal inequalities which are non-binding, we find that

$$\delta_1 = \delta_2 = \lambda^+ = \mu_1^- = 0$$

Moreover, we have $u_1 > 0$ thus

$$-50 \cdot \mu_1^+ + 10 \cdot \mu_1^- + \delta_1 = 0$$

And since $\delta_1 = 0$ we conclude that $\mu_1^+ = \frac{1}{5} \cdot \mu_1^- = 0$.
Moreover, since $u_2 > 0$ we conclude that

$$-50 \cdot \mu_2^+ + 50 \cdot \mu_2^- + \delta_2 = 0$$

And since $\delta_2 = 0$ we conclude that $\mu_2^- = \mu_2^+$ (equal to an arbitrary non-negative constant).
Since $p_1 > 0$ we have that

$$\mu_1^+ - \mu_1^- - \rho_1 + 50 = 0$$

And since $\mu_1^+ = \mu_1^- = 0$, we have that $\rho_1 = 50$.s
Since $p_2 > 0$ we have that

$$\mu_2^+ - \mu_2^- - \rho_2 + 10 = 0$$

Moreover, since $\mu_2^+ = \mu_2^-$, we have that $\rho_2 = 10$.
Finally, from the last KKT condition we have that

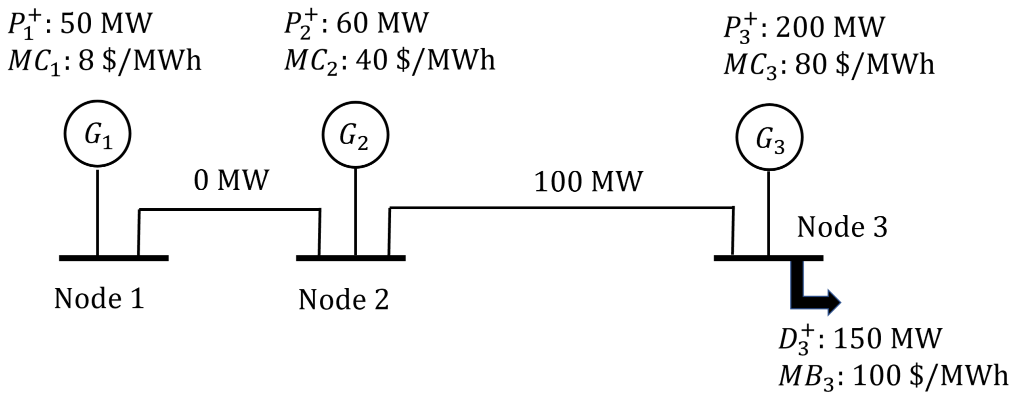
$$\lambda^- = \rho_1 - \rho_2 + \lambda^+ = 50 - 10 + 0 = 40$$

Question 5: Producers are paid their cost, therefore their LOC is 0.

Question 6: If the system operator had the freedom to choose the most profitable flow, it would choose to transfer 10 MW from the cheapest to the most expensive location, thus $\Pi^\star = 10 \cdot (50 - 10) = 400$ \$. Instead, no power is transmitted, thus $\Pi = 0$. Thus, $LOC = 400$ \$ for the transmission system operator, and thus 400 \$ is also the total market LOC.

Problem 7.5: Consider the following optimal power flow problem, which corresponds to the system in figure 7.2.

Figure 7.2: The system of problem 7.5.



$$
\begin{aligned}
p^\star \quad = \quad & \max_{p\geq 0, d\geq 0, r, f} 100 \cdot d_3 - 8 \cdot p_1 - 40 \cdot p_2 - 80 \cdot p_3 \\
(\lambda_1): \quad & r_1 - p_1 = 0 \\
(\lambda_2): \quad & r_2 - p_2 = 0 \\
(\lambda_3): \quad & r_3 + d_3 - p_3 = 0 \\
& p_1 \leq 50 \\
& p_2 \leq 60 \\
& p_3 \leq 200 \\
& d_3 \leq 150 \\
& f_{1-2} = r_1 \\
& f_{2-3} = r_1 + r_2 \\
& f_{1-2} \leq 0 \\
& f_{1-2} \geq 0 \\
& f_{2-3} \leq 100 \\
& f_{2-3} \geq -100 \\
& r_1 + r_2 + r_3 = 0
\end{aligned}
$$

We define the following dual function:

$$
\begin{aligned}
S(\lambda_1, \lambda_2, \lambda_3) \quad = \quad & \max_{p \geq 0, d \geq 0, r, f} 100 \cdot d_3 - 8 \cdot p_1 - 40 \cdot p_2 - 80 \cdot p_3 \\
& -\lambda_1 \cdot (r_1 - p_1) - \lambda_2 \cdot (r_2 - p_2) - \lambda_3 \cdot (r_3 + d_3 - p_3) \\
& p_1 \leq 50 \\
& p_2 \leq 60 \\
& p_3 \leq 200 \\
& d_3 \leq 150 \\
& f_{1-2} = r_1 \\
& f_{2-3} = r_1 + r_2 \\
& f_{1-2} \leq 0 \\
& f_{1-2} \geq 0 \\
& f_{2-3} \leq 100 \\
& f_{2-3} \geq -100 \\
& r_1 + r_2 + r_3 = 0
\end{aligned}
$$

As in the case of the unit commitment problem, lost opportunity cost is a measure of deviation from market equilibrium induced by a given choice of market prices, and is given by the difference $S(\lambda_1, \lambda_2, \lambda_3) - p^\star$.

Question 1: Compute the optimal solution $(p^\star, d^\star, r^\star, f^\star)$ and the maximum possible surplus that the market can generte $p^\star$.

Question 2: Compute the lost opportunity cost given the locational marginal prices $(\lambda_1^\star, \lambda_2^\star, \lambda_3^\star) = (8, 80, 80)$. Exploit the observation that the function $S(\lambda_1, \lambda_2, \lambda_3)$ decomposes into three types of sub-problems for market prices $(\lambda_1, \lambda_2, \lambda_3)$: (i) one profit maximization sub-problem for each producer, (ii) one surplus maximization sub-problem for each consumer, (iii) one surplus maximization sub-problem for the system operator who buys power in cheap locations and sells it in expensive locations.

Question 3: As we analyze in detail in the example of section 5.3.4 of the textbook, zonal pricing distorts the market because it is not aligned with the incentives of market participants. A quantitative measure of this distortion is the lost opportunity cost of zonal prices. Compute the lost opportunity cost of the zonal prices which would emerge if nodes 1 and 2 were merged into a single zone, $(\lambda_1, \lambda_2, \lambda_3) = (40, 40, 80)$.

Solution

Question 1: Unit 1 cannot produce, therefore $p_1^\star = 0$. Thus $r_1^\star = 0$ and $f_{1-2}^\star = 0$. We have that $p_2^\star = 60$ and therefore $r_2^\star = 60$ and $f_{2-3}^\star = 60$. Finally we have that $p_3^\star = 90$ and $d_3^\star = 150$, with $r_3^\star = -60$ and $f_{2-3}^\star = 60$. Thus $p^\star = 100 \cdot 150 - 60 \cdot 40 - 90 \cdot 80 = 15000 - 2400 - 7200 = 5400$.

Question 2: We write the dual function as:

$$S(\lambda_1, \lambda_2, \lambda_3) = S_{G_1}(\lambda_1) + S_{G_2}(\lambda_2) + S_{G_3}(\lambda_3) + S_{L_3}(\lambda_3) + S_N(\lambda_1, \lambda_2, \lambda_3)$$

where

$$\begin{aligned} S_{G_1}(\lambda_1) &= \max_{p_1 \geq 0} (\lambda_1 - 8) \cdot p_1 \\ & \quad p_1 \leq 50 \end{aligned}$$

The profit for $\lambda_1 = 7$ with simple inspection is 0 \$ since any feasible production is optimal.

$$\begin{aligned} S_{G_2}(\lambda_2) &= \max_{p_2 \geq 0} (\lambda_2 - 40) \cdot p_2 \\ & \quad p_2 \leq 60 \end{aligned}$$

The profit for $\lambda_2 = 80$ with simple inspection is $40 \cdot 60 = 2400$ \$ with 60 MW being an optimal solution.

$$\begin{aligned} S_{G_3}(\lambda_3) &= \max_{p_3 \geq 0} (\lambda_3 - 80) \cdot p_3 \\ & \quad p_3 \leq 200 \end{aligned}$$

The profit for $\lambda_3 = 80$ with simple inspection is 0 and any feasible production is optimal.

$$\begin{aligned} S_{L_3}(\lambda_3) &= \max_{d_3 \geq 0} (100 - \lambda_3) \cdot d_3 \\ & \quad d_3 \leq 150 \end{aligned}$$

The surplus for $\lambda_3 = 80$ with simple inspection is $20 \cdot 150 = 3000$ \$ since the optimal consumption is equal to 150 MW.

$$
\begin{aligned}
&S_N(\lambda_1, \lambda_2, \lambda_3) = \max_{r,f} -\lambda_1 \cdot r_1 - \lambda_2 \cdot r_2 - \lambda_3 \cdot r_3 \\
&f_{1-2} = r_1 \\
&f_{2-3} = r_1 + r_2 \\
&r_1 + r_2 + r_3 = 0 \\
&f_{1-2} \leq 0 \\
&f_{1-2} \geq 0 \\
&f_{2-3} \leq 100 \\
&f_{2-3} \geq -100 \\
&r_1 + r_2 + r_3 = 0
\end{aligned}
$$

The system operator achieves a zero profit on line 1-2, since there can be no flow on this line. It also has a zero profit on line 2-3 since prices are equal between nodes 2 and 3.
We thus have that $S(8, 80, 80) = 0 + 0 + 2400 + 3000 + 0 = 5400$.
We note that the sum of each agent surplus equals $p^\star$ and therefore lost opportunity cost is zero. This establishes the fact that locationafl marginal prices are economic equilibrium prices which are aligned in terms of incentives with the optimal dispatch of the system.

Question 3: We have that $S(40, 40, 80) = 8600$, see the solution to problem 2.4 of this manual. Thus the LOC is 3200 \$.

Problem 7.6: Consider a market with the following offers:

- Sell block order (not continuous, all or nothing) 10 MW at a price of 80 \$/MWh
- Buy block order (not continuous, all or nothing) 8 MW at a price of 200 \$/MWh
- Continuous buy order 10 MW at a price of 60 \$/MWh

Compute:

- The optimal order matching and the corresponding social welfare
- The order matching, welfare and market price based on European pricing rules implemented by EUPHEMIA (section 7.3.4), as well as the lost opportunity cost (LOC) for this price, where LOC is computed based on the order matching based on European market clearing (which does not necessarily coincide with socially optimal order matching).
- Integer programming pricing (section 7.3.3), as well as the LOC for this pricing approach, where LOC is computed for the socially optimal order matching.
- The price based on linear programming relaxation (section 7.3.3), as well as the LOC for this pricing approach, where LOC is computed for the socially optimal order matching.

Solution: The optimal matching is as follows:

- Block sell order is accepted for 10 MW
- Block buy order is accepted at 8 MW
- Continuous buy order is accepted for 2 MW

The welfare becomes $200 \cdot 8 + 60 \cdot 2 - 80 \cdot 10 = 920$.
European market clearing seeks a clearing of maximum surplus with a price that is compatible with the market clearing rules. If the two block orders are accepted, then the price must exceed 60 $/MWh due to the continuous buy order which is at the money, but this is not compatible with the block sell order. If the block buy order is rejected but the block sell order is not, then a consumption is cleared which has a valuation that is lower than selling cost, thus it is preferable to not accept any block order, in which case the welfare becomes zero and any price at or above 60 $/MWh is acceptable. At a price of 60 $/MWh, the LOC of the sell block order is (selfish profit – market profit = 0 - 0 = 0), of the buy block order it is (selfish profit - market profit = $140 \cdot 8$ - 0 = 1120) and of the continuous offer (selfish profit – market profit = 0 - 0 = 0). The total LOC is 1120 $.
For integer programming pricing, the price is the one that would emerge if block orders were fixed a priori to their optimal value. The continuous buy

order is the only one which is partially accepted in this case, thus it sets the clearing price, i.e. 60 \$/MWh. At a price of 60 \$/MWh, the LOC of the block sell order is (selfish profit – market profit = 0 - $(60 - 80) \cdot 10 = 200$), the LOC of the block buy order is (selfish profit – market profit = $(200 - 60) \cdot 8 - (200 - 60) \cdot 8 = 0$) and the continuous order LOC is (selfish profit – market profit = 0 - 0 = 0). The total LOC is 200.
For linear programming relaxation pricing, the market price is the one that would emerge if block orders were continuous, i.e. 80 \$/MWh since the sell order would be filled for a positive but not full quantity. At a price of 80 \$/MWh, the LOC of the sell block order is (selfish profit – market profit = 0 - 0 = 0), the LOC of the buy block order is (selfish profit – market profit = $(200 - 80) \cdot 8 - (200 - 80) \cdot 8 = 0$) and the LOC of the continuous order is (selfish profit – market profit = $0 - (60 - 80) \cdot 2 = 40$). The total LOC is 40, lower than that of IP pricing, as the theory predicts.

Chapter 8

Hydrothermal planning

Problem 8.1 A gambler takes a seat at the Casino Royale in order to roll the dice with \$1000 in his pocket. The gambler plans to bet at most once. If he chooses to gamble, the game is played as follows: the gambler picks a number between 1 and 6, the casino rolls the dice, and if the gambler picks the right number then he receives five times his bet, otherwise he loses his bet. Casino Royale is renowned for its quality, and the dice of the game has been through thorough controls to prove that it is fair (with equiprobable results). Compute the value of the stochastic solution (VSS) and the expected value of perfect information (EVPI) for the gambler's problem.

Solution: Since the payoffs of the casino are not fair, the optimal is for the gambler to not play. Thus $SP = 1000$, since he keeps his original wealth.

For the case WS, the gambler knows in advance the outcome of the roll of the dice and therefore always bets his entire wealth, thus he earns $WS = 5000$.

For the case of EEV, the expected value of the roll of the dice is $(1/6) \cdot (1 + 2 + 3 + 4 + 5 + 6) = 3.5$. If the gambler bets 3.5, he always loses. Thus $EEV = 0$.

We therefore have the following:

$$EVPI = WS - SP = 5000 - 1000 = 4000$$
$$VSS = SP - EEV = 1000 - 0 = 1000$$

Problem 8.2: Let us consider the newsvendor problem in its general form:

$$\min_{x,y} C \cdot x - \int_{D=-\infty}^{+\infty} (f(D) \cdot R \cdot y) dD$$
$$y \leq D$$
$$y \leq x$$

Here, x is the number of newspapers that are ordered in the first stage, y is the number of newspapers that are sold in the second stage, C is the price at which the newsvendor buys the newspapers in the first stage, R is the price at which he sells them in the second stage, D is the uncertain demand, and $f(D)$ is the probability density function of the uncertain parameter D.

Question 1: Prove that the optimal order is equal to

$$x = F^{-1}(\frac{R - C}{R})$$

where F is the integral of f, i.e. the cumulative distribution function of demand. Use the fact that the derivative of $\int_{D=-\infty}^{+\infty} h(x, D) dD$ with respect to x is $\int_{D=-\infty}^{+\infty} \frac{\partial_x h(x,D)}{\partial x} dD$ and use the fact that the optimum corresponds to a point where the derivative of total cost with respect to x equals zero.

Question 2: Compute the optimal order x for a uniform distribution between 0 and 1000 newspapers, $R = 4$ \$, and $C = 1$ \$.

Question 3: Compute the expected value of perfect information (EVPI).

Question 4: Compute the value of the stochastic solution (VSS).

Solution

Question 1: The optimal second stage decision as a function of x is $\min(x, D)$. Thus the overall problem can be expressed as

$$\min_{x} C \cdot x - \int_{D=-\infty}^{+\infty} (f(D) \cdot R \cdot \min(x, D)) dD$$

Thus, the derivative of the objective function with respect to x is

$$\begin{aligned}
&C - \int_{D=-\infty}^{+\infty} (f(D) \cdot R) dD = 0 \Rightarrow \\
&C = R \cdot (1 - F(x)) \\
&1 - F(x) = \frac{C}{R} \\
&F(x) = \frac{R - C}{R} \\
&x = F^{-1}(\frac{R - C}{R})
\end{aligned}$$

Question 2: For a uniform distribution we have that the cumulative distribution function of demand is equal to

$$F(x) = \frac{x}{1000}$$

Thus

$$F^{-1}(y) = 1000 \cdot y$$

which implies that

$$x = F^{-1}(\frac{R - C}{R}) = 1000 \cdot (\frac{4 - 1}{4}) = 750$$

Question 3: If we know the demand beforehand, we always choose to order exactly a number of newspapers exactly equal to the demand from the first stage. Thus the cost becomes

$$\begin{aligned}
WS &= \int_{D=-\infty}^{+\infty} (f(D) \cdot (C - R) \cdot D) dD \\
&= \int_{D=0}^{1000} (\frac{1}{1000} \cdot (1 - 4) \cdot D) dD \\
&= \frac{-3}{1000} \cdot \frac{1000^2}{2} = -1500
\end{aligned}$$

If we use the stochastic programming solution then we have

$$
\begin{aligned}
SP &= 1 \cdot 750 - \int_{D=0}^{750} (\frac{3}{1000} \cdot D) dD - \int_{D=750}^{1000} (\frac{3}{1000} \cdot 750) dD \\
&= 750 - \frac{3}{1000} \cdot \frac{1}{2} \cdot 750^2 - \frac{3 \cdot 750 \cdot 250}{1000} \\
&= -656.25
\end{aligned}
$$

We thus have

$$
EVPI = SP - WS = -656.25 - (-1500) = 843.75
$$

Question 4: The expected value of demand is 500. If we order 500 newspapers, we attain a performance of

$$
\begin{aligned}
EEV &= 1 \cdot 500 - \int_{D=0}^{500} (\frac{3}{1000} \cdot D) dD - \int_{D=500}^{1000} (\frac{3}{1000} \cdot 500) dD \\
&= 500 - \frac{3}{1000} \cdot \frac{1}{2} \cdot 500^2 - \frac{3 \cdot 500 \cdot 500}{1000} \\
&= -625
\end{aligned}
$$

We thus have

$$
VSS = EEV - SP = -625 - (-656.25) = 31.25
$$

Chapter 9

Financial instruments

Problem 9.1 Let us consider a risk-averse agent with the following utility function:

$$U(x) = \max(x, 0.5x)$$

where x is the cost that the agent faces. The risk premium RP of a lottery ζ is the amount by which the expected payoff of the agent needs to increase so that the agent becomes indifferent about between the risk of the lottery and $\mathbb{E}[\zeta]$:

$$U(\mathbb{E}[\zeta] + RP) = \mathbb{E}[U(\zeta)].$$

- Compute the risk premium for an agent who faces a game of heads or tails where the agent bets \$1000.
- True/false with justification: the agent prefers to pay \$200 and not bet than playing heads or tails with a bet of \$1000.

Solution

Question 1: The heads or tails lottery has an expected value of $\mathbb{E}[\zeta] = 0$. We are seeking a value RP that solves the following equation:

$$\max(0 + RP, 0) = 0.5 \cdot \max(1000, 500) + 0.5 \cdot \max(-1000, -500).$$

For $RP \geq 0$, we have

$$RP = 500 - 250 = 250.$$

Question 2: True. We can substitute into the utility function or observe that the \$200 is less than the risk premium of the agent.

Problem 9.2 Let us consider an electricity supplier that wishes to procure 10 MWh of electricity for delivery in a year from now. We assume that the price of electricity in a year from now is distributed as follows:

$$40 + 0.1 \cdot i \text{ \$/MWh},$$

with a probability of 10% for each outcome $i = 1, \ldots, 10$.

1. What is the conditional value at risk for a risk level of 15% for the supplier, ($CVaR_{0.85}$)?
2. Let us assume that the supplier wishes to eliminate its risk, by procuring a forward contract today for future delivery of electricity in a year from now. At what price will the contract be available in the market, assuming that agents in the market are risk neutral (thus equation (9.1) holds)?
3. What cash flows does the forward contract imply today, and at the expiration of the forward, if the price of electricity turns out to be 100 \$/MWh at the expiration date and if the supplier indeed consumes 10 MWh at the expiration date?
4. What is the conditional value at risk $CVaR_{0.85}$ for the supplier if he enters a forward contract? Is he better of with or without the contract, based on the CVaR criterion?

Solution

Question 1: The probability distrbution is: 50 \$/MWh with a probability of 10%, ..., 140 \$/MWh with a probability of 10%. The 15% worst outcomes have an expected value of: $(2/3) \cdot 140 + (1/3) \cdot 130 = 136.67\$/MWh$. Thus, the exposure of the supplier for 10 MWh is \$1366.7.

Question 2: The forward contract price qill equal the expected price of electricity, thus

$$(1/10) \cdot 50 + (1/10) \cdot 60 + \ldots + (1/10) \cdot 140 = 0.1 \cdot 5 \cdot 180 = 90 \text{ \$/MWh}.$$

Question 3: Today the supplier pays (10 MWh) · (90 \$/MWh) = \$900 for buying the contract.
At the expiration date the supplier (i) pays (100 \$/MWh)·(10 MWh) = \$1000 for consuming electricity in real time and (ii) is paid (100 \$/MWh)·(10 MWh) = \$1000 from settling the forward contract.

Question 4: The payoff of the supplier if he engages in a forward contract is (A) what he pays today, i.e. \$900, (B) what he is paid for settling the forward contract (uncertain) and (C) what he pays for consuming in real time(also uncertain). But we observe that term C cancels out term C, thus we are left with term A with certainty. Thus we have that $CVaR_{0.85}$ for the supplier is \$900. This cost is lower than the \$1366.7 that is computed in question 1, thus the supplier is better off with the forward contract.

Chapter 10

Demand response

Problem 10.1: Consider a time of use (ToU) pricing model for a system which has a constant demand curve (thus there is no distinction between peak hours and base-load hours) with the following characteristics:

- Inverse demand curve: $MB(d) = 1000 - 0.1 \cdot d$
- Short-term marginal cost of production: $MC = 6.5$ \$/MWh
- Long-term marginal cost of production: $MI = 25$ \$/MWh

What is the optimal investment (x), the optimal demand (d), the optimal production (p), and the ToU prices (ρ) in this system?
The KKT analysis shows that the valuation of consumers at the optimal level of investment and production must equal the sum of short-term and long-term marginal cost:

$$
\begin{aligned}
&MI + MC = MB(d) \\
&\Rightarrow 25 + 6.5 = 1000 - 0.1 \cdot d \\
&\Rightarrow d = 9685 \text{ MW} = p = x
\end{aligned}
$$

The price becomes $\rho = 31.5$ \$/MWh, i.e. just the sum of MI and MC.

Chapter 11

Generation capacity expansion

Problem 11.1: True or false: competitive markets with marginal pricing result in higher consumer bills than the good old days when regulated monopolies where consumers were charged at average cost.

Solution: False. We have proven in chapter 11 that the competitive long-run equilibrium is one in which all technologies exactly break even. Thus, the revenue collected by consumers is exactly equal to that of charging consumers for the sum of variable plus fixed cost over all technologies.

Problem 11.2: Levelized cost of energy for a technology g is defined in exercise 11.4 as:

$$LCOE = \frac{IC_g \cdot x_g + \sum_{j=1}^{m} \Delta T_j \cdot MC_g \cdot p_{gj}}{\sum_{j=1}^{m} \Delta T_j \cdot p_{gj}}.$$

Consider a technology g with investment cost IC_g and marginal cost MC_g. Denote κ as the fraction of time when market prices are greater than or equal to the marginal cost of a technology, and $\bar{\lambda}$ as the average price over these periods. Prove that an investor would choose to invest in technology g if

$$\bar{\lambda} \geq LCOE.$$

Solution: The price duration curve essentially determines when technology g should operate. Let us consider, without loss of generality, an investment in a unit (1 MW) of generation capacity. Denote also equivalently the set of time intervals as $J = \{1, \ldots, m\}$. Given a market price λ_j for time interval

j, a technology will operate in this interval at full capacity if $\lambda_j > MC_g$, and at partial or full capacity if $\lambda_j = MC_g$, but in the last case the short-term profits are the same (and equal to zero) whether the technology operates in partial or full capacity.
Denote the fraction of time that a technology operates as:

$$\kappa = \sum_{j \in J : \lambda_j \geq MC_g} \Delta T_j.$$

Then, LCOE can be expressed equivalently as follows:

$$\begin{aligned} LCOE &= \frac{IC_g \cdot x_g + \sum_{j=1}^{m} \Delta T_j \cdot MC_g \cdot p_{gj}}{\sum_{j=1}^{m} \Delta T_j \cdot p_{gj}} \\ &\simeq \frac{IC_g \cdot 1 + \sum_{j \in J : \lambda_j \geq MC_g} \Delta T_j \cdot MC_g \cdot 1}{\sum_{j \in J : \lambda_j \geq MC_g} \Delta T_j \cdot 1} \\ &= \frac{IC_g}{\kappa} + MC_g. \end{aligned}$$

The approximate equality in the second line is because of the assumption that the technology operates at full capacity when it is at the money.
Long-term profits can be expressed as

$$\begin{aligned} \sum_{j \in J} \Delta T_j \cdot \mu_{gj} - IC_g &= \sum_{j \in J} \Delta T_j \cdot (\lambda_j - MC_g) \cdot p_{gj} - IC_g \\ &= \sum_{j \in J} \Delta T_j \cdot (\lambda_j - MC_g) \cdot p_{gj} - IC_g \\ &= \sum_{j \in J : \lambda_j \geq MC_g} \Delta T_j \cdot (\lambda_j - MC_g) \cdot 1 - IC_g \end{aligned}$$

Denote the conditional average of prices for those periods when the unit is producing as $\bar{\lambda}$. Then:

$$\begin{aligned} \bar{\lambda} &= \frac{\sum_{j \in J : \lambda_j \geq MC_g} \Delta_j \cdot \lambda_j}{\sum_{j \in J : \lambda_j \geq MC_g} \Delta_j} \\ &= \frac{\sum_{j \in J : \lambda_j \geq MC_g} \Delta_j \cdot \lambda_j}{\kappa} \end{aligned}$$

Plugging back to the previous equality, the long-term profits can be re-expressed as

$$\begin{aligned}\sum_{j\in J}\Delta T_j \cdot \mu_{gj} - IC_g &= \sum_{j\in J:\lambda_j\geq MC_g}\Delta T_j\cdot(\lambda_j - MC_g) - IC_g \\ &= \kappa\cdot\bar{\lambda} - \kappa\cdot MC_g - IC_g\end{aligned}$$

The investment criterion of the technology, as explained in the textbook, is to invest if long-term profits are non-negative:

$$\begin{aligned}&\kappa\cdot\bar{\lambda} - \kappa\cdot MC_g - IC_g \geq 0 \Leftrightarrow \\ &\bar{\lambda} - MC_g - \frac{IC_g}{\kappa} \geq 0 \Leftrightarrow \\ &\bar{\lambda} \geq LCOE.\end{aligned}$$

Problem 11.3 Consider a technology with an investment cost of $IC_g = 5$ \$/MWh and a marginal cost of $MC_g = 80$ \$/MWh. Suppose that the price duration curve in the market is as follows:

- The price is 150 \$/MWh for 10% of the time.
- The price is 85 \$/MWh for 40% of the time.
- The price is 60 \$/MWh for 30% of the time.
- The price is 25 \$/MWh for 20% of the time.

Is the technology worth investing in? What is the LCOE of the technology?

Solution: The short-term profits of the generator for 1 MW of built capacity are:

$$(150-80)\cdot 0.1 + (85-80)\cdot 0.4 = 7 + 2 = 9\$/MWh.$$

The long-term profits of the generator, on the other hand, are 5 \$/MWh. Therefore, the technology is worth investing into.
The LCOE is:

$$LCOE = \frac{5 + 0.1\cdot 80 + 0.4\cdot 80}{0.1+0.4} = 32.4\$/MWh$$

Another way to reach the same conclusion is by using the result of exercise 11.2. The fraction of time that price exceeds the marginal cost of the technology is $\kappa = 0.5$. The average price during these periods is

$$\bar{\lambda} = 0.2 \cdot 150 + 0.8 \cdot 85 = 98\$/MWh$$

Thus, we have that $\bar{\lambda} > LCOE$. According to the result of exercise 11.2, the technology is worth investing in.

Problem 11.4: Consider the capacity expansion planning problem of example 11.2, where the technologies that are available for investment are as follows:

- Coal with a fuel cost of 25 \$/MWh and an investment cost of 16 \$/MWh
- Natural gas with a fuel cost of 80 \$/MWh and an investment cost of 5 \$/MWh
- Nuclear with a fuel cost of 6.5 \$/MWh and an investment cost of 32 \$/MWh
- Oil with a fuel cost of 160 \$/MWh and an investment cost of 2 \$/MWh

Let us consider a load shedding "technology" with zero investment cost and a "fuel cost" that is equal to the value of lost load ($VOLL$). What should the $VOLL$ be in \$/MWh so that there are 3 hours of load shedding at the optimal expansion planning solution?

1. $VOLL > 6000$ \$/MWh
2. 4540 \$/MWh $\leq VOLL \leq$ 6000 \$/MWh
3. $VOLL < 4540$ \$/MWh

Solution: The correct answer is option 2.
From screening curve analysis we have that load shedding is preferable than the next alternative (peak technology, i.e. oil) when:

$$VOLL \cdot (\frac{3}{8760}) \leq I_n + MC_n \cdot (\frac{3}{8760}) \Rightarrow VOLL \leq 6000 \text{ \$/MWh}$$

Moreover, from screening curve analysis we know that at 4 hours it is preferable to resort to the peak technology instead of curtailment, thus

$$VOLL \cdot (\frac{4}{8760}) \geq I_n + MC_n \cdot (\frac{3}{8760}) \Rightarrow VOLL \geq 4540 \text{ \$/MWh}$$

Alternatively, from footnote 44 of the textbook we can conclude that

$$I_n \simeq VOLL \cdot (\frac{3}{8760}) \Rightarrow VOLL = \frac{8760}{3} \cdot 2 = 5840 \text{ \$/MWh}$$

Thus, again, we conclude that the correct option is 2.

Problem 11.5: Consider three technologies with the following characteristics:

- T1: investment cost of 15 \$/MWh, fuel cost of 30 \$/MWh
- T2: investment cost of 30 \$/MWh, fuel cost of 5 \$/MWh
- T3: investment cost of 5 \$/MWh, fuel cost of 150 \$/MWh

Let us assume that the load duration curve of the system is described by the following horizontal slices:

- Base load horizontal slice: 3000 MW for 100% of the time
- Medium load horizontal slice: 5000 MW for 50% of the time
- Peak load horizontal slice: 2000 MW for 5% of the time

Question 1: Compute the optimal technology mix.

Question 2: Compute the optimal production of each technology.

Question 3: Compute the market equilibrium prices.

Solution

Question 1: The load duration curve has the following breakpoints.
From T2 to T1:

$$30 + 5 \cdot f = 15 + 30 \cdot f \Rightarrow f = 0.6$$

From T1 to T3:

$$15 + 30 \cdot f = 5 + 150 \cdot f \Rightarrow f = 0.0833$$

Thus, we assign:

- the base load to T2, thus 3000 MW
- the medium load to T1, thus 5000 MW
- the peak load to T3, thus 2000 MW

Question 2: The optimal production is as follows, given the investments in question 1 (where the indices base, medium and peak now refer to vertical load slices):

$$\begin{aligned}
p_{T2,base} &= p_{T2,medium} = p_{T2,peak} = 3000 \text{ MW} \\
p_{T1,base} &= 0 \text{ MW}, p_{T2,medium} = p_{T2,peak} = 5000 \text{ MW} \\
p_{T3,base} &= p_{T3,medium} = 0 \text{ MW}, p_{T3,peak} = 2000 \text{ MW}
\end{aligned}$$

Question 3: Equilibrium prices are computed from the fact that each technology that invests must break even. For T3 we have:

$$5 = 0.05 \cdot (\rho_{peak} - 150) \Rightarrow \rho_{peak} = 250 \text{ \$/MWh}$$

For T1 we have:

$$15 = 0.45 \cdot (\rho_{medium} - 30) + 0.05 \cdot (250 - 30) \Rightarrow \rho_{medium} = 38.89 \text{ \$/MWh}$$

For T2 we have:

$$30 = 0.5 \cdot (\rho_{base} - 5) + 0.45 \cdot (38.89 - 5) + 0.05 \cdot (250 - 5) \Rightarrow \rho_{base} = 10 \text{ \$/MWh}$$

Problem 11.6: The **capacity factor** CF is the ratio between the production of a unit and is capacity, $p_{ij} = CF \cdot x_i$. The **levelized cost of energy** (LCOE) λ is defined in problem 11.4 of the textbook as the ratio of the total cost of production by a unit i and the total energy produced:

$$\lambda = \frac{IC_i \cdot x_i + \sum_{j=1}^{m} \Delta T_j \cdot MC_i \cdot p_{ij}}{\sum_{j=1}^{m} \Delta T_j \cdot p_{ij}}.$$

True/false with justification: Given investment cost IC_i and marginal cost MC_i for a technology i, $LCOE$ is increasing with respect to the capacity factor of the unit.

Solution: False. We have the following:

$$\begin{aligned}
\lambda &= \frac{IC_i \cdot x_i + \sum_{j=1}^{m} \Delta T_j \cdot MC_i \cdot p_{ij}}{\sum_{j=1}^{m} \Delta T_j \cdot p_{ij}} \\
&= \frac{IC_i \cdot x_i + \sum_{j=1}^{m} \Delta T_j \cdot MC_i \cdot CF \cdot x_i}{\sum_{j=1}^{m} \Delta T_j \cdot CF \cdot x_i} \\
&= \frac{IC_i + \sum_{j=1}^{m} \Delta T_j \cdot MC_i \cdot CF}{\sum_{j=1}^{m} \Delta T_j \cdot CF} \\
&= \frac{IC_i + MC_i \cdot CF \cdot \sum_{j=1}^{m} \Delta T_j}{CF \cdot \sum_{j=1}^{m} \Delta T_j} \\
&= \frac{IC_i + MC_i \cdot CF}{CF} \\
&= \frac{IC_i}{CF} + MC_i
\end{aligned}$$

Thus, λ is decreasing with respect to CF.

Chapter 12

Beyond electricity

Problem 12.1: Formulate the Russian taxation model as a linear complementarity problem and replicate the results of problem 12.3.

Solution: The model is expressed as follows. The quantity adjustment model of the European market can be expressed as the European demand function. The quantity adjustment model of the Asian market can be expressed as the Asian supply function. The quantity adjustment model of the Russian monopoly is expressed as:

$$\max_{p \geq 0}(F(p) - C - t) \cdot p$$

The tax is denoted as t. The function $F(p)$ maps the production of the monopolist to the price paid by Europe. It is essentially the inverse of the function $D_R(\lambda)$ in example 12.3. For the numbers used specifically in this example we have

$$F(p) = 34.542 - 0.00213 \cdot p$$

The KKT conditions of the Russian profit maximization problem are expressed as:

$$0 \leq p \perp -F(p) + C + t - F'(p) \cdot p \geq 0$$

The code is available in the course website.

Problem 12.2: In textbook problem 12.3 we found that a tax on Russian gas can backfire for Europeans, if that tax is chosen to be equal to 10 \$/mcf.

Re-run the model of problem 12.3 for a varying level of taxes (for a Russian marginal cost of 11.18 \$/mcf, as computed in the solution of the problem), and find the tax that is optimal for European consumers.
Gros proves analytically (in equation 23 of his report) that the optimal tariff is given by

$$t_{OT} \cdot (4 - r \cdot d_e) = 2 \cdot (R - c - r \cdot D) + r \cdot d_e \cdot (R + c)$$

The parameters of the above equation are given as follows:

$$\begin{aligned} r &= 1/(d_e + s_a) \\ R &= \frac{D + S_a}{d_e + s_a} \end{aligned}$$

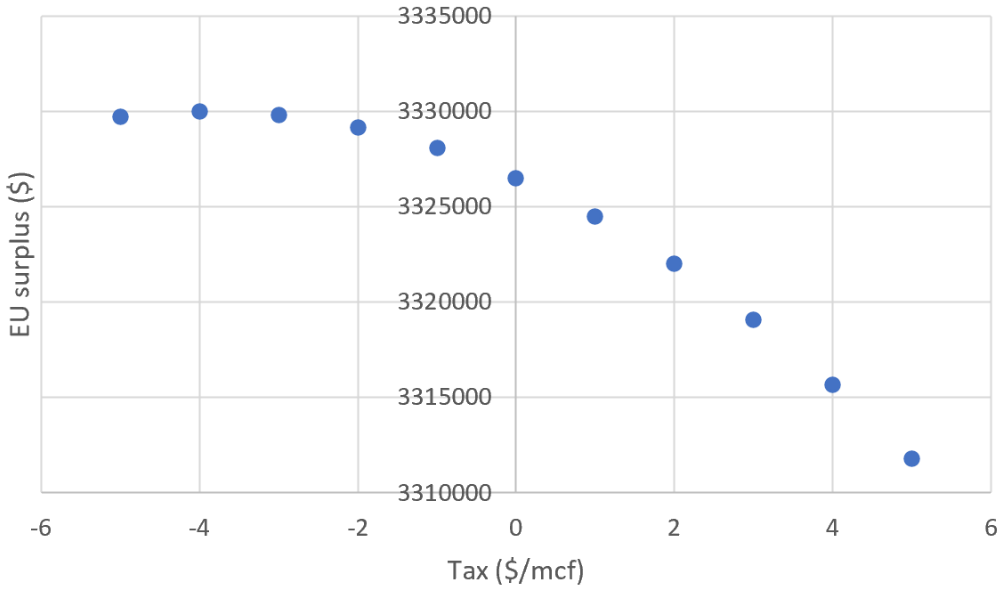
where the parameters D and d_e are the parameters of the European demand curve $(D - d_e \cdot \lambda)$ and S_a and s_a are the parameters of the Asian supply curve $(-S_a + s_a \cdot \lambda)$. The parameter c is the marginal cost of Gazprom. Do your results coincide with Gros' analysis?

Solution: We run the model of problem 12.1 of this manual for various choices of tax, from -5 to 5, with an increment of 1. We plot the results in figure 12.1.
From example 12.3 of the textbook, we have (i) the following calibration of the European demand curve for natural gas: $15277.3 - 31.82 \cdot \lambda$; (ii) the following calibration for total non-Russian supply: $-907.60 + 436.73 \cdot \lambda$; (iii) a marginal cost of 11.18.
Thus, the parameters in Gros' models are as follows:

$$\begin{aligned} D &= 15277.3 \\ S_a &= 907.6 \\ d_e &= 31.82 \\ s_a &= 436.73 \\ r &= 1/(d_e + s_a) = 1/(31.82 + 436.73) = 1/468.55 = 0.00213 \\ R &= \frac{D + S_a}{d_e + s_a} = \frac{15277.3 + 907.6}{31.82 + 436.73} = \frac{16184.9}{468.55} = 34.542 \end{aligned}$$

Thus, the equation derived by Gros can be stated as follows, for the specific

Figure 12.1: EU welfare as a function of tax imposed to Russia in problem 12.1.



numbers of our example:

$$t_{OT} \cdot (4 - r \cdot d_e) = 2 \cdot (R - c - r \cdot D) + r \cdot d_e \cdot (R + c) \Rightarrow$$
$$t_{OT} \cdot (4 - 0.00213 \cdot 31.82) = 2 \cdot (34.542 - 11.18 - 0.00213 \cdot 15277.3) +$$
$$3.932 \cdot t_{OT} = -15.258 \Rightarrow$$
$$t_{OT} = \frac{-15.258}{3.932} = -3.88\$/\text{mcf}$$

The results of Gros coincide with those of the numerical model. The policy conclusion is interesting, because it actually points out that it is best not to apply a tax, but rather to apply a subsidy.

Problem 12.3: Consider a market with an inverse demand function $MB(d) = 1000 - 40d$, and an inverse supply function $MC(p) = 10 \cdot p$.

Question 1: What is the equilibrium market price, and the traded quantity?

Question 2: If we introduce a tax of 50 \$, what is the new equilibrium price in the market and the traded quantity?

Question 3: What is the deadweight loss from the introduction of the tax?

Question 4: Which side of the market absorbs the majority of the tax?

Solution

Question 1: The equilibrium price is given by solving the following equation:

$$1000 - 40 \cdot q = 10 \cdot q \Rightarrow q = 20 \Rightarrow \lambda = 200.$$

Question 2: Given a tax of \$50, the new equilibrium is given from the solution of the following equation:

$$1000 - 40 \cdot q = 10 \cdot q + 50 \Rightarrow q = 19 \Rightarrow \lambda^b = 240, \lambda^s = 190.$$

Question 3: The deadweight loss is:

$$\begin{aligned}&\int_{q=19}^{20}(MB(q)-MC(q))dq=\\&\int_{q=19}^{20}(1000-40\cdot q-10\cdot q)dq=\\&1000\cdot(20-19)-50\cdot 0.5\cdot(20^2-19^2)=25\end{aligned}$$

Question 4: We have the following:

$$\lambda^b-\lambda=40, \lambda-\lambda^s=10.$$

So the majority of the tax is absorbed by consumers, which is not surprising since they have a steeper inverse demand function.

Problem 12.4

Question 1: True/false with justification: The marginal benefit curve $MB(d) = A\cdot d^{-\frac{1}{\epsilon}}$ results in an isoelastic demand.

Question 2: Suppose that the global oil market is isoelastic with an elasticity of -5 (meaning that a 1% increase in prices results in a 5% decrease in demand) and historically it has been observed that a price of 80 $/barrel results in a consumption of 30 billion barrels annually. Express the function $MB(d)$.

Solution

Question 1: True. We have that

$$\text{Elasticity}=\frac{dQ/Q}{d\lambda/\lambda}=\frac{\lambda}{Q}\frac{1}{d\lambda/dQ}=\frac{Ad^{-\frac{1}{\epsilon}}}{d}\frac{1}{A(-\frac{1}{\epsilon})d^{-\frac{1}{\epsilon}-1}}=-\epsilon.$$

Question 2: We have $\epsilon=5$, and

$$80=A\cdot 30^{-\frac{1}{5}}\Rightarrow A=157.9.$$

Problem 12.5: Let us assume that the global oil market is perfectly competitive, and that the supply function is expressed as follows:

$$P(\lambda) = 33.85 + 0.023 \cdot \lambda \text{ billion barrels},$$

where λ (in \$/barrel) is the price of oil.

Question 1: What is the market equilibrum (oil price and annual trade of oil) given the inverse demand function

$$MB(d) = 2400 \cdot d^{-1}.$$

Question 2: Due to the war in the Middle East, the marginal cost curve of oil is shifted upwards by 20 \$/barrel. Express the new supply function $P(\lambda)$.

Question 3: Compute the market equilibrium (oil price and annual trade of oil) after the Middle East crisis.

Solution

Question 1: We first express the marginal cost curve of the market:

$$MC(p) = -\frac{33.85}{0.023} + \frac{1}{0.023}p = -1471.7 + 43.47 \cdot p.$$

We then equate marginal cost to marginal benefit:

$$2400 \cdot p^{-1} = -1471.7 + 43.47p \Rightarrow 43.47 \cdot p^2 - 1471.4 \cdot p - 2400 = 0.$$

We solve the second order system:

$$p = \frac{1471.4 \pm \sqrt{1471.4^2 - 4 \cdot 43.47 \cdot (-2400)}}{2 \cdot 43.47} = \frac{1471.4 \pm 1607.2}{2 \cdot 43.47}.$$

The two solutions are 35.41 and -1.56, we only keep the positive solution. The market clearing price is:

$$MB(35.41) = 67.78 \text{ \$/barrel}.$$

Question 2: The new marginal cost curve is

$$MC(p) = 20 - 1471.4 + 43.47 \cdot p = -1451.7 + 43.47 \cdot p.$$

Inverting the function, we have

$$P(\lambda) = 33.40 + 0.023 \cdot \lambda.$$

Question 3: We repeat the procedure of question 1 with the new marginal cost curve:

$$2400 \cdot p^{-1} = -1451.7 + 43.47 \cdot p.$$

The solutions are 34.97 and -1.58. We only keep the positive solution, and derive a market price of 2400/34.97=68.63 \$/barrel.

Problem 12.6: Solve Hotelling's model for $\epsilon = -1$, $r = 10\%$, $C = 0.1\$/\text{unit}$, $S = 10$ units, and $T = 2$ periods. What is the price of the finite energy resource at each period? What is the consumption at each period?

Solution: We can write down four equations that lead us to a second order equation that solves the problem. Concretely:

$$\begin{aligned}
&\lambda_1 = d_1^{-1/\epsilon} = \frac{1}{d_1} \\
&\lambda_2 = d_2^{-1/\epsilon} = \frac{1}{d_2} \\
&d_1 + d_2 = S = 10 \\
&\lambda_2 - C = (1+r) \cdot (\lambda_1 - C) \Rightarrow \lambda_2 - 0.1 = 1.1 \cdot (\lambda_1 - 0.1)
\end{aligned}$$

Substituting the first three equations into the fourth one, we have

$$\begin{aligned}
&\frac{1}{d_2} - 0.1 = 1.1 \cdot (\frac{1}{10 - d_2} - 0.1) \Rightarrow \frac{1 - 0.1 \cdot d_2}{d_2} = 1.1 \frac{1 - 1 + 0.1 \cdot d_2}{10 - d_2} \Rightarrow \\
&(1 - 0.1 \cdot d_2) \cdot (10 - d_2) = 1.1 \cdot d_2 \cdot (0.1 d_2) \Rightarrow \\
&10 - d_2 - d_2 + 0.1 \cdot d_2^2 = 0.11 \cdot d_2^2 \Rightarrow \\
&10 - 2 \cdot d_2 + 0.1 \cdot d_2^2 = 0.11 \cdot d_2^2 \Rightarrow 0.01 \cdot d_2^2 + 2 \cdot d_2 - 10 = 0
\end{aligned}$$

We have two solutions to this second order equation, from which we only maintain the non-negative one: $d_2 = 4.881$. From this we conclude that $d_1 = 5.119$. And the market prices are $\lambda_1 = 0.195$ \$/unit and $\lambda_2 = 0.205$ \$/unit. Substituting back to Hotelling's rule, we confirm that the rule holds and that we have therefore correctly solved the system:

$$\begin{aligned}
&\lambda_2 - C = 0.205 - 0.1 = 0.105 \\
&(1+r) \cdot (\lambda_1 - C) = 1.1 \cdot (0.195 - 0.1) = 0.105
\end{aligned}$$

Problem 12.7: Consider a simple model of perfect competition for the global natural gas market, which consists of two regions, the USA and Europe. We represent European demand using an inverse demand function that is expressed as follows:

$$MB(Q) = 1000 - 10 \cdot Q \text{ \$/barrel},$$

where Q (in billion barrels) is the amount of barrels consumed in Europe. Let us assume that the European marginal cost curve is:

$$MC_E(Q) = 40 + Q \text{ \$/barrel},$$

where Q (in billion barrels) is the number of barrels produced in Europe.

Question 1: What is the price of natural gas in the European market, if the European market is isolated from the rest of the world?

Question 2: Let us assume that the USA is in a position to export to Europe at a marginal cost of

$$MC_A(Q) = 30 + 2Q \text{ \$/barrel}$$

What is the price of natural gas in Europe? How does it compare to the answer of question 1? How much natural gas does the USA export to Europe?

Question 3: A linear approximation of the global natural gas market is described as follows:

$$\begin{aligned}
& \max_{p_A, p_E, d, f_{A-E}} && V \cdot d - MC_A \cdot p_A - MC_E \cdot p_E \\
(\lambda_E): & && d - p_E - f_{A-E} = 0 \\
(\lambda_A): & && f_{A-E} - p_A = 0 \\
(\mu_A): & && p_A \leq P_A \\
(\mu_E): & && p_E \leq P_E \\
(\nu): & && d \leq D \\
& && p \geq 0, d \geq 0
\end{aligned}$$

Here, p_A is the production of natural gas in the US, p_E is the production of natural gas in Europe, f_{A-E} is the transportation of natural gas from the

USA to Europe (with a negative sign implying a transportation of gas from Europe to the US), d is the consumption of natural gas in Europe, V is the valuation of natural gas from European consumers, MC_A is the marginal cost of production of natural gas in the US, MC_E is the marginal cost of natural gas in Europe, P_A is the capacity of natural gas in the US, P_E is the capacity of natural gas in Europe, and D is the European load for natural gas.
Express the KKT conditions of the model. What is the relation between the US price λ_A and the European price λ_E?

Solution

Question 1: This is merely the intersection of the supply and demand functions, thus we solve the following equation:

$$1000 - 100 \cdot Q = 40 + Q$$

which yields $Q = 9.505$ billion barrels and $\lambda = 40 + Q = 49.505$ \$/barrel.

Question 2: At economic equilibrium the marginal cost of each market is equal to the marginal benefit of consumers, thus we solve the following system of equations:

$$\begin{aligned} 30 + 2 \cdot p_A &= 1000 - 100 \cdot (p_A + pE) \\ 30 + 2 \cdot p_A &= 40 + p_E \end{aligned}$$

From the second equation:

$$p_E = 2p_A - 10.$$

And substituting back to the first equation:

$$30+2\cdot p_A = 1000-100\cdot(p_A+2\cdot p_A-10) = 1000-300\cdot p_A+1000 = 2000-300\cdot p_A.$$

Thus $p_A = 6.523$ and $p_E = 3.046$. The equilibrium market price is $40 + p_E = 43.046$ \$/barrel. The price drops, because the US contributes to covering demand. The US exports 6.523 billion barrels in Europe.

Question 3: The KKT conditions are described as follows:

$$
\begin{aligned}
& p_E + f_{A-E} = d \\
& p_A = f_{A-E} \\
& 0 \leq \mu_A \perp p_A \leq P_A \\
& 0 \leq \mu_E \perp p_E \leq P_E \\
& 0 \leq \nu \perp d \leq D \\
& 0 \leq p_A \perp MC_A - \lambda_A + \mu_A \geq 0 \\
& 0 \leq p_E \perp MC_E - \lambda_E + \mu_E \geq 0 \\
& 0 \leq d \perp -V + \nu + \lambda_E \geq 0 \\
(f_{A-E}) : \quad & -\lambda_E + \lambda_A = 0
\end{aligned}
$$

The prices are equal, as a consequence of the last KKT condition.

Problem 12.7: Suppose that European demand for natural gas is 15000 billion cubic feet per year, and that the year is split into two semesters, the first semester (months March-August) with a demand of 5000 billion cubic feet per year and the second semester (months September-February) with a demand of 10000 billion cubic feet. The supply curve per semester, in billion cubic feet, for the rest of the world except Russia is

$$P(\lambda) = -450 + 220 \cdot \lambda$$

where the price of natural gas λ is measured in \$/MWh. Le us assume that Russian supply of natural gas in Europe is inelastic and equal to 200 billion cubic feet in the first semester and 800 billion cubic feet in the second semester.

Question 1: What is the equilibrium market price for the first and the second semester if we assume that Europe cannot store natural gas?

Question 2: On December 2024 a five-year contract for transporting Russian natural gas through Ukraine expired. Ukraine chose not to renew the contract, in order to obstruct the financing of the war between Russia and Ukraine. We represent this evolution in our model by assuming that Russian natural gas supply to Europe is interrupted for both semesters. Compute the new equilibrium prices for each semester.

Question 3: Let us assume that the ability to store gas in Europe is so large that it leads to the equalization of the price of natural gas between the first and the second semester of the year (which corresponds to the least-cost way to cover the needs of Europe for natural gas). Compute the equilibrium price of the market of question 2 (i.e. after the supply of Russian natural gas has been interrupted) at each semester of the year when we have storage. How many billion cubic feet of natural gas are stored from the first semester to the second?

Question 4: The newly elected president of the United States who sits in office in a few weeks aims at imposing taxes on the export of natural gas from the US to the rest of the world. Let us assume that the new policy of the US is represented in the model as an increase in the non-Russian supply function (inverse of $P(\lambda)$) by 3 \$/MWh. Compute the new equilibrium price under the conditions of question 1.3 (unlimited storage and interruption of Russian natural gas) as well as the quantity of natural gas that is transferred from the first semester to the second one.

Solution

Question 1: For the first semester the price is the solution of the following equation:

$$-450 + 220 \cdot \lambda + 200 = 5000$$

The solution for the first semester is

$$\lambda = \frac{5250}{220} = 23.86\$/MWh$$

For the second semester the price is the solution of the following equation:

$$-450 + 220 \cdot \lambda + 800 = 10000$$

The solution for the second semester is

$$\lambda = \frac{9650}{220} = 43.86\$/\text{MWh}$$

Question 2: For the first semester the price is the solution of the following equation:

$$-450 + 220 \cdot \lambda = 5000$$

The solution for the first semester is

$$\lambda = \frac{5450}{220} = 24.77\$/\text{MWh}$$

For the second semester the price is the solution of the following equation:

$$-450 + 220 \cdot \lambda = 10000$$

The solution for the second semester is

$$\lambda = \frac{10450}{220} = 43.86\$/\text{MWh}$$

Question 3: We seek the solution to the following system of equations:

$$\begin{aligned}
&(-450 + 220 \cdot \lambda_1) + (-450 + 220 \cdot \lambda_2) = 15000 \\
&\lambda_1 = \lambda_2
\end{aligned}$$

Substituting the second equation into the first, we have

$$\begin{aligned}
&(-450 + 220 \cdot \lambda_1) + (-450 + 220 \cdot \lambda_1) = 15000 \Rightarrow \\
&-900 + 440 \cdot \lambda_1 = 15000 \Rightarrow \\
&\lambda_1 = \frac{15900}{440} = 36.14\$/\text{MWh}
\end{aligned}$$

We compute the production of the second semester as

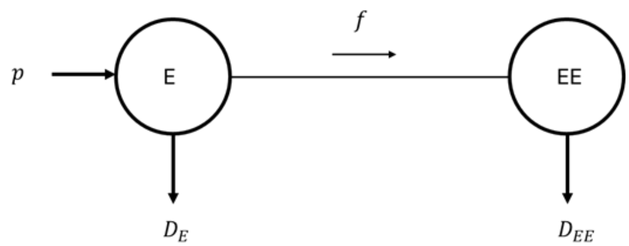
$$-450 + 220 \cdot \lambda_1 = -450 + 220 \cdot 36.14 = 7500.8 \text{ billion cubic feet}$$

Subtracting from these the 5000 billion cubic feet of demand of the first semester, we conclude that 2500.8 billion cubic feet are transferred from the first to the second semester.

Question 4: Since the demand of Europe is inelastic, the market price simply incrases by 3 $/MWh and the quantity of natural gas that is transferred between the two semesters remains the same.

Problem 12.8: Greece is a point of import of natural gas in Europe. This, in contrast to claims that have recently been made in the news, in itself does not secure that Greece has access to lower electricity prices. In order to

Figure 12.2: Graphical representation of the economic model of problem 12.8.



understand this, let us analyze a model of the natural gas market of Greece which is represented in figure 12.2 and is expressed as follows:

$$\min_{p,f} A \cdot p^2 + B \cdot B$$
$$(\lambda_E): \quad D_E + f - p = 0$$
$$(\lambda_{EE}): \quad D_{EE} - f = 0$$
$$p \geq 0$$

Here, p corresponds to the quantity of LNG that reaches Greece from the rest of the world, $A \cdot p^2 + B \cdot p$ is the total cost of imported LNG, f are the exports of natural gas to the rest of Europe, D_E is the demand for natural gas in Greece and D_{EE} is the demand for natural gas in the rest of Europe.

Question 1: Extract the KKT conditions of the problem. Reminder: type 3 KKT conditions for non-linear programs are generalized by taking the *derivative* with respect to the corresponding primal variable (instead of the coefficient of the variable, as is the case in linear programs).

Question 2: True/false with justification: Greece enjoys a lower price for natural gas.

Solution

Question 1: The KKT conditions are expressed as follows:

$$D_E + f = p$$
$$D_{EE} = f$$
$$0 \leq p \perp 2 \cdot A \cdot p + B - \lambda_E \geq 0$$
$$\lambda_E - \lambda_{EE} = 0$$

Question 2: False. The last condition of question 1 establishes that the prices will be equal.

Problem 12.9: Let us consider a linear inverse supply function which is expressed analytically as $MC(p) = A + B \cdot p$ and a linear inverse demand function which is expressed analytically as $MB(d) = C - D \cdot d$. The parameters A, B, C and D are positive.

Question 1: Multiple choice with justification: The amoun t of trade in a market with a tax t is equal to

1. $\frac{C-A+t}{B+D}$
2. $-\frac{A}{B}$
3. $\frac{C-A-t}{B+D}$
4. $D + \frac{t}{C}$

Question 2: True/false with justification: Reduced elasticity of supply (greater B) implies a greater volume of trade at the market equilibrium.

Question 3: True/false with justification: Reduced elasticity of supply (greater B) implies a greater price at which the good is traded at the market equilibrium.

Solution

Question 1: The economic equilibrium in this market is given by the following conditions:

$$\lambda_s = A + B \cdot p$$
$$\lambda_b = C - D \cdot d$$
$$\lambda_b = \lambda_s + t$$
$$p = d$$

Substituting the third and fourth condition into the first and second, we have

$$C - D \cdot p = A + B \cdot p + t \Rightarrow (B + D) \cdot p = C - A - t \Rightarrow p = \frac{C - A - t}{B + D}$$

The correct choice is thus option 3.

Question 2: False. The volume of trade is decreasing with respect to B, because the denominator of the expression of question 1 increases.

Question 3: True. The volume of trade d is reduced, thus the price $\lambda_b = C - D \cdot d$ increases, and since $\lambda_s = \lambda_b - t$ increases we conclude that λ_s also increases.

Problem 12.10: Let us consider the model of a dominant firm in an electricity market with the following data:

- Marginal cost of dominant firm: $MC_d = 10$ \$/MWh
- Inverse demand function: $MB(q) = 900 - q$ \$/MWh
- Marginal cost of fringe producers: $MC_f(q) = q$ \$/MWh

Question 1: Compute the inverse net demand function $MB_N(q)$ which describes the market price that is required for leaving q MWh of demand as a market share for the monopoly.

Question 2: What is the economic equilibrium (monopoly production, fringe production, consumption, market price) when the dominant firm behaves as a monopoly, i.e. when it maximizes $MB_N(q) \cdot q - MC_d \cdot q$? What is the social surplus of the market? What is the profit of the monopoly?

Question 3: Repeat the calculations of questions 1 and 2 for the following inverse demand function: $MB(q) = 1800 - 2 \cdot q$ \$/MWh. Does the market price increase? Does the profit of the monopoly increase?

Solution

Question 1: We solve the following system of equations in order to express price λ as a function of $p_d = d - p_f$:

$$\lambda = 900 - d$$
$$\lambda = p_f$$

Adding both sides:

$$\lambda + \lambda = 900 - (d - p_f) \Rightarrow 2 \cdot \lambda = 900 - p_d$$

Thus

$$MB_N(q) = 450 - 0.5 \cdot q.$$

Question 2: The problem solved by the monopoly is:

$$\max_q (450 - 0.5 \cdot q) \cdot q - 10 \cdot q \Rightarrow$$
$$\max_q -0.5 \cdot q^2 + 440 \cdot q$$

The derivative becomes equal to zero at the solution of the following equation:

$$-q + 440 = 0 \Rightarrow q = 440.$$

We thus have:

- Monopoly production: $p_d = 440$ MWh
- Market price: $\lambda = 450 - 0.5 \cdot p_d = 450 - 220 = 230$ \$/MWh
- Fringe production: $p_f = \lambda = 230$ MWh
- Consumption: $d = p_d + p_f = 440 + 230 = 670$ MWh

And we indeed confirm that

$$MB(d) = 900 - 670 = 230 \text{ \$/MWh}$$

The social surplus is:

$$\int_{x=0}^{d} MB(x)dx - MC_d \cdot d - \int_{x=0}^{p_f} MC_f(x)dx$$
$$= 900 \cdot d - 0.5 \cdot d^2 - 10 \cdot p_d - 0.5 \cdot p_f^2$$
$$900 \cdot 670 - 0.5 \cdot 670^2 - 10 \cdot 440 - 0.5 \cdot 230^2 = 347700$$

The monopoly profit is:

$$(\lambda - MC_d) \cdot p_d = (230 - 10) \cdot 440 = 96800\$$$

Question 3: We solve the following system of equations in order to express price λ as a function of $p_d = d - p_f$:

$$\lambda = 1800 - 2 \cdot d$$
$$\lambda = p_f$$

Adding both sides:

$$\lambda + 2 \cdot \lambda = 1800 - 2 \cdot (d - p_f) \Rightarrow 3 \cdot \lambda = 1800 - 2 \cdot p_d$$

Thus

$$MB_N(q) = 600 - \frac{2}{3} \cdot q.$$

The problem solved by the monopoly is:

$$\max_q (600 - \frac{2}{3} \cdot q) \cdot q - 10 \cdot q \Rightarrow$$
$$\max_q -\frac{2}{3} \cdot q^2 + 590 \cdot q$$

The derivative becomes equal to zero at the solution of the following equation:

$$-\frac{4}{3}q + 590 = 0 \Rightarrow q = 442.5.$$

We thus have:

- Monopoly production: $p_d = 442.5$ MWh
- Market price: $\lambda = 600 - \frac{2}{3} \cdot p_d = 600 - \frac{2}{3} 442.5 = 305$ \$/MWh
- Fringe production: $p_f = \lambda = 305$ MWh
- Consumption: $d = p_d + p_f = 442.5 + 305 = 747.5$ MWh

And we indeed confirm that

$$MB(d) = 1800 - 2 \cdot 747.5 = 305 \text{ \$/MWh}$$

The social surplus is:

$$\int_{x=0}^{d} MB(x)dx - MC_d \cdot d - \int_{x=0}^{p_f} MC_f(x)dx$$
$$= 1800 \cdot d - 0.5 \cdot 2 \cdot d^2 - 10 \cdot p_d - 0.5 \cdot p_f^2$$
$$1800 \cdot 747.5 - 0.5 \cdot 2 \cdot 747.5^2 - 10 \cdot 442.5 - 0.5 \cdot 305^2 = 735806.25$$

The monopoly profit is:

$$(\lambda - MC_d) \cdot p_d = (305 - 10) \cdot 442.5 = 130538\$$$

Price increases, and so does the profit of the monopoly.

Problem 12.11: Consider a population of consumers wih a demand function for oil (in billion barrels per year)

$$D(\lambda) = 40 - 0.05 \cdot \lambda$$

where λ is the price of oil in \$/barrel. Derive the inverse demand function $MB(d)$. Compute the consumer benefit (in \$ per year) which results from consuming 2 billion barrels per year.

Solution: The inverse demand curve is derived by inverting the demand curve:

$$d = 40 - 0.05 \cdot \lambda \Rightarrow$$
$$\lambda = 20 \cdot (40 - d) \Rightarrow$$
$$MB(d) = 800 - 2 \cdot d \text{ \$/barrel}$$

Consumer benefit resulting from the consumption of 2 billion barrels per year is equal to

$$\begin{aligned}
&\int_{x=0}^{2}(800 - 20 \cdot x)dx = \\
&800 \cdot 2 - 20 \cdot \frac{2^2}{2}\frac{\$}{\text{barrel}} \times \text{billion barrels} = \\
&1600 - 40 \text{ billion } \$ = 1560 \cdot 10^9\$
\end{aligned}$$

Careful, we would be incorrect to compute the solution by integrating

$$\int_{x=0}^{2\cdot 10^9}(800 - 20 \cdot x)dx$$

which would yield a negative result, because in order to compute consumer benefit in this way we would also need to adjust the slope of the inverse demand function.
It would also be incorrect to compute

$$\int_{x=0}^{2}(800 - 20 \cdot x)dx = 1560\$$$

because in this way we have ignored the fact that the units of measurement of valuation (in \$/barrel) are multiplied with the units of measurement of quantity (in billion barrels, not barrels).

Problem 12.12: We recall that the KKT conditions for a non-linear program

$$\begin{aligned}
(NC): \quad & \max_{x \geq 0, y} f(x,y) \\
(\lambda_i): \quad & g_i(x,y) \leq 0, i = 1, \ldots, m \\
(\mu_i): \quad & h_i(x,y) = 0, i = 1, \ldots, p
\end{aligned}$$

are expressed as follows and generalize the linear case:

$$\begin{aligned}
\text{Type 1}: \quad & h(x,y) = 0 \\
\text{Type 2}: \quad & 0 \leq \lambda \perp g(x,y) \leq 0 \\
\text{Type 3}: \quad & 0 \leq x \perp \lambda^T \nabla_x g(x,y) + \mu^T \nabla_x h(x,y) - \nabla_x f(x,y) \geq 0 \\
\text{Type 4}: \quad & \lambda^T \nabla_y g(x,y) + \mu^T \nabla_y h(x,y) - \nabla_y f(x,y) = 0
\end{aligned}$$

where $\nabla_x f(x, y)$ is the partial derivative of a function f with respect to x.

Question 1: Express the KKT conditions of a population of oil consumers with an inverse demand function $MB(d) = 500 - 10 \cdot d$ \$/barrel who face a market price of λ \$/barrel and decide to consumer a quantity of d billion barrels per year:

$$\begin{aligned} &\max_{d} \int_{x=0}^{d} MB(x)dx - \lambda \cdot d \\ &d \geq 0 \end{aligned}$$

Question 2: What is the solution of the KKT system (optimal consumption) for a market price of $\lambda = 100$ \$/barrel?

Question 3: What is the solution of the KKT system (optimal consumption) for a market price of $\lambda = 600$ \$/barrel?

Question 4: Let us assume that the population of producers is expressed by a marginal cost function $MC(p) = 20 + 2 \cdot p$ \$/barrel, where p is given in billion barrels. The competitive market equilibrium is expressed as a social surplus maximization problem:

$$\begin{aligned} &\max_{p,d} \int_{x=0}^{d} MB(x)dx - \int_{x=0}^{p} MC(x)dx \\ (\lambda): \quad &d - p = 0 \\ &p \geq 0, d \geq 0 \end{aligned}$$

Express the KKT conditions of the problem.

Question 5: Compute the competitive market equilibrium, i.e. the solution of the KKT system λ (in \$/barrel), p (in billion barrels), d (in billion barrels) of question 4s.

Question 6: What is the consumer benefit (in \$)? What is the producer surplus (in \$)? What is the production cost (in \$)? What is the producer profit (in \$)? What is the social surplus (in \$)?

Solution

Question 1: We have that $\int_{x=0}^{d} MB(x)dx = 500 \cdot d - 5 \cdot d^2$. We only have type 3 conditions:

$$0 \leq d \perp -500 + 10 \cdot d + \lambda \geq 0$$

Question 2: Assuming that the optimal consumption is positive, we have that

$$-500 + 10 \cdot d + \lambda = 0 \Rightarrow -500 + 10 \cdot d + 100 = 0 \Rightarrow d = 40 \text{ billion barrels.}$$

Indeed, this decision is a solution to the KKT system.

Question 3: Assuming that the optimal consumption is zero, we have that

$$-500 + 10 \cdot d + \lambda = 0 \Rightarrow -500 + 10 \cdot 0 + 100 \geq 0$$

Thus, the KKT system is indeed satisfied by zero consumption, therefore zero consumption is indeed an optimal reaction.

Question 4: The type 1 conditions are

$$d - p = 0$$

The type 3 conditions are:

$$0 \leq d \perp -MB(d) + \lambda \geq 0 \Leftrightarrow 0 \leq d \perp -(500 - 10 \cdot d) + \lambda \geq 0$$
$$0 \leq p \perp MC(p) - \lambda \geq 0 \Leftrightarrow 0 \leq p \perp 20 + 2 \cdot p - \lambda \geq 0$$

Question 5: We can simply compute the intersection of the supply and demand curves, and confirm hat this solves the KKT system of question 4. Concretely, the intersection is given by the following equation:

$$20 + 2 \cdot x = 500 - 10 \cdot x \Rightarrow 12 \cdot x = 480 \Rightarrow x = 40$$

Thus $p = d = 40$ billion barrels per year and $\lambda = 20 + 2 \cdot 40 = 500 - 10 \cdot 40 = 100$ \$/barrel. We confirm that these values solve the KKT system.

Question 6
Consumer benefit: $\in_{x=0}^{40} MB(x)dx = 500 \cdot 40 - 5 \cdot 40^2 = 12000$ billion \$

Consumer surplus: $\int_{x=0}^{40} MB(x)dx - 100 \cdot 40 = 8000$ billion \$
Production cost: $\int_{x=0}^{40} MC(x)dx = 20 \cdot 40 + 40^2 = 2400$ billion \$
Producer profit: $100 \cdot 40 - \int_{x=0}^{40} MC(x)dx = 1600$ billion \$
Social surplus: $1600 + 8000 = 9600$ billion \$

Problem 12.13: Let us consider an electricity market with 1000 MW of inelastic demand, fringe/perfectly competitive producers with a linear marginal cost curve which is equal to 0 \$/MWh at 0 MW and 500 \$/MWh at 1000 MW.

Question 1: Compute the marginal cost curve and the supply curve of the fringe/perfectly competitive producers.

Let us assume that there is a dominant firm in the market with a marginal cost of 100 \$/MWh and an installed capacity of 1000 MW.

Question 2: Compute the economic equilibrium (production of dominant firm, production of fringe producers, market clearing price) under conditions of perfect competition. What is the profit of the fringe producers? What is the profit of the dominant firm?

Question 3: Compute the economic equilibrium (production of dominant firm, production of fringe producers, market clearing price) in the case where the dominant firm behaves as a monopoly, as in chapter 12.3 of the textbook, i.e. sets quantity so as to maximize profit and where fringe producers react as followers. What is the profit of the fringe producers? What is the profit of the dominant firm?

Question 4: True/false with justification:

- Fringe producers lose market share when the dominant firm behaves as a monopoly.
- Fringe producers increase profitability when the dominant firm behaves as a monopoly.

Solution

Question 1: The supply curve of fringe producers is

$$S(\lambda) = 2 \cdot \lambda.$$

And the marginal cost curve is

$$MC(p) = 0.5 \cdot p.$$

Question 2: In the competitive equilibrium, the dominant firm produces a positive quantity pd below its technical maximum, we thus have

$$100 = 0.5 \cdot pf \Rightarrow pf = 200 \text{ MW}$$

Thus, the dominant firm produces $pd = 800$ MW.
The market clearing price is 100 \$/MWh. Indeed, the marginal cost curve of question 1 at 200 MW is 100 \$/MWh.
The profit of the fringe producers is

$$\Pi f = \int_{x=0}^{200} (100 - 0.5 \cdot x) dx = 100 \cdot 200 - 0.25 \cdot 200^2 = 10000\$$$

The profit of the dominant firm is 0 \$.

Question 3: Given a market price λ, the demand served by the monopoly is $1000 - 2 \cdot \lambda$. Inverting the equation, we compute the price that the monopoly us paid as a function of the quantity that it offers to the market:s

$$\lambda = \frac{1000 - pd}{2}.$$

Thus the monopoly decides on the quantity that it makes available to the market by solving the following problem:

$$\max_{pd} (\frac{1000 - pd}{2}) \cdot pd - 100 \cdot pd = 400 \cdot pd - 0.5 \cdot pd^2$$

Setting the derivative equal to zero, we conclude that the optimal production of the monopoly is $pd = 400$ MW. This leads to a markt price of

$$\lambda = \frac{1000 - pd}{2} = \frac{1000 - 400}{2} = 300 \text{ \$/MWh}$$

This in turn leads to a production by fringe producers that is equal to $pf =$ 600 MW. We indeed confirm that $pd + pf = 400 + 600 = 1000$ MW.
The profit of the monopoly is

$$\Pi d = (300 - 100) \cdot 400 = 80000\$.$$

The profit of the fringe producers is

$$\Pi f = \int_{x=0}^{600} (300 - 0.5 \cdot x) dx = 300 \cdot 600 - 0.25 \cdot 600^2 = 90000\$$$

Question 4
The first claim is false since the fringe production increases from 200 MW to 600 MW. Which makes sense since the dominant firm reduces its production and demand is inelastic.
The second claim is true since the fringe producer profit increases from 10000 $ to 90000 $. Which also makes sense since the dominant firm increases its market share and sells at a higher price.

Problem 12.14: Let us consider a firm with a marginal cost function $MC(Q)$ and a market with an inverse demand curve $\lambda(Q)$. We recall that demand elasticity is defined as the percent decrease in demand Q for a 1% increase in the market price λ, $\epsilon = \frac{dQ/Q}{d\lambda/\lambda}$, and is therefore a non-positive quantity, i.e. $\epsilon \leq 0$.

Question 1: Multiple choice with justification: The percent increase in the market price above competitive levels, $\frac{\lambda - MC(Q)}{\lambda}$, is equal to:

1. $\frac{\lambda - MC(Q)}{\lambda} = \frac{1}{\epsilon - 1}$
2. $\frac{\lambda - MC(Q)}{\lambda} = -\frac{1}{\epsilon + 1}$
3. $\frac{\lambda - MC(Q)}{\lambda} = -\frac{1}{\epsilon}$

Question 2: True/false with justification: lower demand elasticity (lower $|\epsilon|$) implies greater opportunities for the firm to exploit its dominant position (greater $\frac{\lambda - MC(Q)}{\lambda}$).

Question 3: Consider a market with an inverse demand function $\lambda(Q) = 50 \cdot d^{-0.5}$ and a monopoly with a marginal cost of 90 \$/MWh. Compute the market clearing price. How does the clearing price change for an inverse demand function $\lambda(Q) = 100 \cdot d^{-0.5}$?

Solution

Question 1: The profit maximization problem of the monopolist is expressed as:

$$\max_q \lambda(Q) \cdot Q - TC(Q)$$

To find where the derivative becomes zero, we proceed as follows:

$$\begin{aligned}
&\lambda'(Q) \cdot Q + \lambda(Q) - MC(Q) = 0 \Rightarrow \\
&\lambda(Q) - MC(Q) = -\lambda'(Q) \cdot Q \Rightarrow \\
&\frac{\lambda(Q) - MC(Q)}{\lambda(Q)} = -\frac{\lambda'(Q) \cdot Q}{\lambda(Q)} = -\frac{\frac{d\lambda}{dQ} \cdot Q}{\lambda} = -\frac{\frac{d\lambda}{\lambda}}{\frac{dQ}{Q}} = -1/\epsilon
\end{aligned}$$

So the correct choice is option 3. The left-hand side of this expression is referred to in economics as the **Lerner index**.

Question 2: From question 1, lower $|\epsilon|$ implies greater $1/|\epsilon|$ and since $\epsilon \leq 0$ we have that $-1/\epsilon$ becomes more positive. Thus the claim is true, the Lerner index increases.

Question 3: Demand is iso-elastic with an elasticity of $\epsilon = -2$. From question 1, the Lerner index is

$$\frac{\lambda - 90}{\lambda} = 1/2 \Rightarrow 2 \cdot \lambda - 180 = \lambda \Rightarrow \lambda = 180 \text{ \$/MWh}.$$

Price does not change in the second case, since demand remains iso-elastic with the same elasticity.

Chapter 13

A brief introduction to linear programming

Problem 13.1: Multiple choice with justification: Consider the following linear programming problem:

$$\begin{aligned} p^\star &= \min_x 7x \\ (\lambda): &\quad x \leq 2 \end{aligned}$$

The dual program:

- Option A: Has an optimal solution with a bounded objective function value $d^\star$ equal to $p^\star$
- Option B: Is infeasible
- Option C: Has an optimal solution that can be arbitrarily large

Solution: The correct choice is B.
The dual program is:

$$\begin{aligned} &\max_\lambda 2\lambda \\ &\lambda = 7 \\ &\lambda \leq 0 \end{aligned}$$

Thus the dual program is infeasible.

Problem 13.2: Compute the dual of the following linear programs, the primal optimal solution (the answer "does not exist" is acceptable), the dual optimal solution (the answer "does not exist" is acceptable), the optimal value of the primal objective function $p^\star$ (the answer "does not exist" is not acceptable, it is a value in $\mathbb{R} \cup \{\pm\infty\}$) and the optimal value of the dual objective function $d^\star$ (the answer "does not exist" is not acceptable, it is a value in $\mathbb{R} \cup \{\pm\infty\}$).

Question 1:

$$\begin{aligned} & \min_{x_1,x_2} 2x_1 + 3x_2 \\ (\mu): \quad & x_1 + 4x_2 = 8 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Question 2:

$$\begin{aligned} & \min_{x_1,x_2} 2x_1 + 3x_2 \\ (\mu): \quad & x_1 + 4x_2 = 8 \\ & x_1 \leq 0, x_2 \leq 0 \end{aligned}$$

Question 3:

$$\begin{aligned} & \min_{x_1,x_2} 2x_1 + 3x_2 \\ (\mu): \quad & x_1 + 4x_2 = 8 \\ & x_1 \geq 0, x_2 \leq 0 \end{aligned}$$

Solution
Question 1: The dual is:

$$\begin{aligned} & \max_{\mu} 8\mu \\ (x_1): \quad & \mu \leq 2 \\ (x_2) \quad & 4\mu \leq 3 \end{aligned}$$

The primal optimal solution is $x_1^\star = 0$, $x_2^\star = 2$, and the objective function value is $p^\star = 6$.
The dual optimal solution is $\mu^\star = 3/4$, and the objective function value is $d^\star = 6$.

Question 2: The dual is:

$$\begin{aligned} & \max_{\mu} 8\mu \\ (x_1): \quad & \mu \geq 2 \\ (x_2) \quad & 4\mu \geq 3 \end{aligned}$$

There is no primal optimal solution because the primal problem is infeasible. The optimal objective function value of the primal is $p^\star = +\infty$.
There is no dual optimal solution because the dual problem is unbounded. The optimal objective function value of the dual is $d^\star = +\infty$.

Question 3: The dual is:

$$\begin{aligned} & \min_{\mu} 8\mu \\ (x_1): \quad & \mu \geq 2 \\ (x_2) \quad & 4\mu \leq 3 \end{aligned}$$

There is no primal optimal solution because the primal problem is unbounded. The optimal objective function value of the primal is $p^\star = +\infty$.
There is no dual optimal solution because the dual problem is infeasible. The optimal objective function value of the dual is $d^\star = +\infty$.

Problem 13.3: Express the following problems as linear programs, or conclude that this is not possible:

A) $\min(3, 6, 8.2)$

B) $\max(3, 6, 8.2)$

C) $\min_x |x|$

D) $\min_x |x|, 0 \leq x \leq 2$

E) $\max_x |x-2|, -10 \leq x \leq 10$

F) $\min_x -|x-2|, -10 \leq x \leq 10$

G) $\min_x |x-2|, -10 \leq x \leq 10$

Solution
A) The problem is expressed as:

$$\begin{aligned}
&\min_{\theta} \theta \\
&\theta \geq 3 \\
&\theta \geq 6 \\
&\theta \geq 8.2
\end{aligned}$$

B) The problem is expressed as:

$$\begin{aligned}
&\max_{\theta} \theta \\
&\theta \leq 3 \\
&\theta \leq 6 \\
&\theta \leq 8.2
\end{aligned}$$

C) The problem is expressed as:

$$\begin{aligned}
&\min_{x,\theta} \theta \\
&\theta \geq x \\
&\theta \geq -x
\end{aligned}$$

D) The problem is expressed as:

$$\min_{x,\theta} \theta$$
$$\theta \geq x$$
$$\theta \geq -x$$
$$0 \leq x \leq 2$$

E) It is not expressed as a linear program, because he problem is not convex.

F) It is not expressed as a linear program, because he problem is not convex.

G) The problem is expressed as:

$$\min_{x,\theta} \theta$$
$$\theta \geq x - 2$$
$$\theta \geq -(x - 2)$$
$$-10 \leq x \leq 10$$

Chapter 14

The direct current power flow

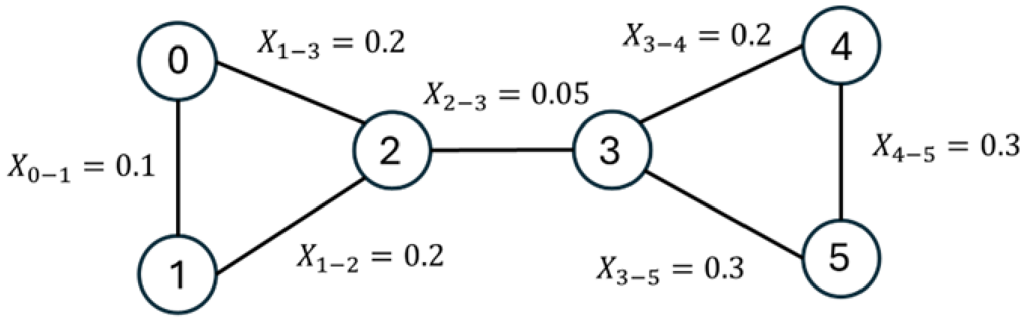
Problem 14.1: The susceptances of the network of figure 14.1 are given in per unit values. Compute all the PTDF coeffcients $PTDF_{(i-j,n)}$ for every line $(i-j) \in \{(0-1),(1-2),(2-3),(3-4),(4-5),(3-5)\}$ and every node $n \in \{1,2,3,4,5\}$, where node 0 is considered the hub node.

Solution: For a parallel connection of two passive elements with susceptances X_1 and X_2, the power that flows through each of the elements is the solution to the following system:

$$
\begin{aligned}
&P_1 + P_2 = 1 \\
&P_1 = \frac{X_2}{X_1} P_2 \Rightarrow P_1 = \frac{X_2}{X_1}(1 - P_1) \Rightarrow (1 + \frac{X_2}{X_1})P_1 = \frac{X_2}{X_1} \Rightarrow \\
&P_1 = \frac{X_2}{X_1 + X_2}, P_2 = \frac{X_1}{X_1 + X_2}
\end{aligned}
$$

The key observation is that, due to the radial connection of the two sub-networks, we can analyze each of the sub-networks 0-1-2 and 3-4-5 separately.

Figure 14.1: The network of exercise B.1.



For node 1:

$$PTDF_{0-1,1} = -\frac{0.4}{0.5} = -0.8$$
$$PTDF_{0-2,1} = -\frac{0.1}{0.5} = -0.2$$
$$PTDF_{1-2,1} = \frac{0.1}{0.5} = 0.2$$
$$PTDF_{2-3,1} = 0$$
$$PTDF_{3-4,1} = 0$$
$$PTDF_{4-5,1} = 0$$
$$PTDF_{3-5,1} = 0$$

For node 2:

$$PTDF_{0-1,2} = -\frac{0.2}{0.5} = -0.4$$
$$PTDF_{0-2,2} = -\frac{0.3}{0.5} = -0.6$$
$$PTDF_{1-2,2} = -\frac{0.2}{0.5} = -0.4$$
$$PTDF_{2-3,2} = 0$$
$$PTDF_{3-4,2} = 0$$
$$PTDF_{4-5,2} = 0$$
$$PTDF_{3-5,2} = 0$$

For node 3:

$$PTDF_{0-1,3} = PTDF_{0-1,2} = -0.4$$
$$PTDF_{0-2,3} = PTDF_{0-2,2} = -0.6$$
$$PTDF_{1-2,3} = PTDF_{1-2,2} = -0.4$$
$$PTDF_{2-3,3} = -1$$
$$PTDF_{3-4,3} = 0$$
$$PTDF_{4-5,3} = 0$$
$$PTDF_{3-5,3} = 0$$

For node 4:

$$PTDF_{0-1,4} = PTDF_{0-1,2} = -0.4$$
$$PTDF_{0-2,4} = PTDF_{0-2,2} = -0.6$$
$$PTDF_{1-2,4} = PTDF_{1-2,2} = -0.4$$
$$PTDF_{2-3,4} = -1$$
$$PTDF_{3-4,4} = -\frac{0.6}{0.8} = -0.75$$
$$PTDF_{4-5,4} = \frac{0.2}{0.8} = 0.25$$
$$PTDF_{3-5,4} = -\frac{0.2}{0.8} = -0.25$$

For node 5:

$$PTDF_{0-1,5} = PTDF_{0-1,2} = -0.4$$
$$PTDF_{0-2,5} = PTDF_{0-2,2} = -0.6$$
$$PTDF_{1-2,5} = PTDF_{1-2,2} = -0.4$$
$$PTDF_{2-3,5} = -1$$
$$PTDF_{3-4,5} = -\frac{0.3}{0.8} = -0.375$$
$$PTDF_{4-5,5} = -\frac{0.3}{0.8} = -0.375$$
$$PTDF_{3-5,5} = -\frac{0.5}{0.8} = -0.625$$

Chapter 15

Dynamic programming

Problem 15.1: Consider the first step of the L-shaped algorithm applied to the following problem:

$$\min_{x,y} 2 \cdot x - 0.5 \cdot (3 \cdot y_1 + 3 \cdot y_2)$$
$$y_1 \leq x$$
$$y_2 \leq x$$
$$y_1 \leq 5$$
$$y_2 \leq 10$$
$$x, y \geq 0$$

The master problem is expressed as:

$$\min_{x,\theta} 2 \cdot x + \theta$$
$$\theta \geq -1000$$
$$x \geq 0$$

Question 1: Why do we include the first inequality in the master problem?

Question 2: What is the optimal solution of the master problem?

The slave problem for the first scenario is expressed as:

$$
\begin{aligned}
& \min_{y} -3 \cdot y_1 \\
(\pi): \quad & y_1 \leq x \\
(\mu): \quad & y_1 \leq 5 \\
& y_1 \geq 0
\end{aligned}
$$

Question 3: Express the dual of the above problem for $x = 0$. What is the primal optimal solution y_1? Propose a dual optimal solution for π and μ. What is the objective function value at the optimal solution?
Question 4: Conduct the same analysis for the second scenario, and compute a primal-dual optimal pair.

Question 5: Compute the optimality cut of the master problem using the solution of questions 3 and 4.

Question 6: Establish that the inequality $\theta \geq -1000$ of the master problem is valid (i.e. -1000 is indeed a lower bound of the expected cost of the second scenario).

Solution

Question 1: So that the solution of the master problem does not become unbounded.

Question 2: The optimal solution is $\theta = -1000$ and $x = 0$.

Question 3: The dual of the slave problem is (with $x = 0$):

$$
\begin{aligned}
& \max_{\pi,\mu} x\pi + 5\mu \\
(y_1): \quad & \pi + \mu \leq -3 \\
& \pi \leq 0, \mu \leq 0
\end{aligned}
$$

The optimal (and unique feasible) solution of the primal problem is $y_1 = 0$. A non-unique solution to the dual problem is $\mu = 0, \pi = -3$. The objective function value is 0.

Question 4: The slave problem is

$$
\begin{aligned}
&\min_{y} -3 \cdot y \\
&(\pi) : y_2 \leq x \\
&(\mu) : y_2 \leq 10 \\
&y_2 \geq 0
\end{aligned}
$$

The dual of the slave problem (with $x = 0$) is:

$$
\begin{aligned}
&\max_{\pi,\mu} x\pi + 10\mu \\
(y_2): \quad &\pi + \mu \leq -3 \\
&\pi \leq 0, \mu \leq 0
\end{aligned}
$$

The optimal (and unique feasible) solution is $y_2 = 0$. A (non-unique) dual optimal solution is $\mu = 0$, $\pi = -3$. The objective function value is 0.

Question 5:
The optimality cut is

$$\theta \geq 0.5 \cdot (5 \cdot 0 - 3 \cdot x) + 0.5 \cdot (10 \cdot 0 - 3 \cdot x) = -3 \cdot x$$

Note that the dual multipliers in questions 3 and 4 could have been obtained from the KKT conditions as well. A subtle point of attention however arises in this case: the KKT conditions for maximization problems, as expressed in the textbook, can indeed be interpreted as sensitivities. This can indeed by confirmed in proposition 2.8 by the fact that the relaxed constrains are *subtracted* from the Lagrangian function. On the other hand, the KKT conditions for minimization problems, if taken by just replacing objective function derivatives with minus objective function derivatives in the type 3 and 4 KKT conditions, lead to dual multipliers that are the sensitivities of maximization problems of minus the original objective function, thus equal to *minus* the sensitivity of the original minimization problem. And since optimality cuts for cutting plane methods require sensitivity, and the evaluation of the original slave objective (not minus the objective), the construction of the optimality cuts requires taking *minus* the inner product of the dual vector from KKT conditions and the respective right-hand side vector.
As a case in point, try deriving the KKT conditions of the slave. You will find that the dual solution that you obtain (denoted (π^{KKT}, μ^{KKT})) becomes

$\mu^{KKT} = 0, \pi^{KKT} = 3$. Thus, the optimality cut is derived as *minus* the inner product of this dual vector and the right-hand sides of the slave problem constraints.

Question 6: The tendency of the slave problem is to increase y as much as possible. Given the bounds $y_1 \leq 5$ and $y_2 \leq 10$, we conclude that the value function cannot become lower than

$$0.5 \cdot (-3 \cdot 5 - 3 \cdot 10) = -22.5$$

Chapter 16

Integer Programming

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